Quantum Information Theory (SS'06)

Problem Set No 2 (20 scores)
emission: 11.05.06; absorption: 25.05.06

In this set you recollect your memory on the spin-1/2 quantum mechanics ...

▷ Aufgabe 1 (Qubitology) (13 scores)

Recall that for the qubit the Pauli spin operator $\hat{\vec{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is of paramount importance. Its cartesian components obey angular momentum commutation relations

$$[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\hat{\sigma}_z\,, \qquad \text{with } xyz \text{ cyclic}\,, \tag{1}$$

and the specific spin-$\frac{1}{2}$ anti-commutation relations

$$\{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij}\hat{1}\,, \qquad i,j = x,y,z\,. \tag{2}$$

where $\hat{1}$ is the identity operator on $\mathcal{H}_{\text{qubit}} \simeq \mathbb{C}^2$, which we shall occasionally denote $\hat{\sigma}_0 \equiv \hat{1}$.

Acting in a two-dimensional Hilbert space, the Pauli operators admit a representation in terms of hermitian 2×2 matrices. Adopting the convention

$$|\uparrow_z\rangle \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |\downarrow_z\rangle \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3}$$

for the eigenvectors of $\hat{\sigma}_z$, the standard matrix representation reads

$$\hat{\sigma}_0 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{\sigma}_x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_y \mapsto \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{\sigma}_z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4}$$

Matrix representations are quite useful – their two-dimensional layout is better adapted to human parallel processing than the serial layout of the abstract notation ...

(a) Let $\vec{a}$ be a (real) spatial unit vector, $|\vec{a}| = 1$, and denote

$$\hat{\sigma}_a := \vec{a} \cdot \hat{\vec{\sigma}} \tag{5}$$

the cartesian component of $\hat{\vec{\sigma}}$ in $\vec{a}$-direction. Confirm that $\hat{\sigma}_a$ is self adjoint. Furthermore

$$\text{tr}\hat{\sigma}_a = 0\,, \tag{6}$$

and

$$\hat{\sigma}_a^2 = \hat{1}\,. \tag{7}$$

Conclude that the spectrum of $\hat{\sigma}_a$ is given $\{+1, -1\}$.

(b) Confirm the master identity

$$\exp\{i\alpha\hat{\sigma}_a\} = \cos(\alpha)\hat{1} + i\sin(\alpha)\hat{\sigma}_a\,. \tag{8}$$

(c) Denote $|\uparrow_a\rangle, |\downarrow_a\rangle$ the eigenvectors of $\hat{\sigma}_a$. Confirm the spectral representation

$$\hat{\sigma}_a = |\uparrow_a\rangle\langle\uparrow_a| - |\downarrow_a\rangle\langle\downarrow_a| \,. \tag{9}$$

Express $|\uparrow_a\rangle, |\downarrow_a\rangle$ in terms of a linear combination of $|\uparrow_z\rangle, |\downarrow_z\rangle$.

(d) Let $\vec{a}$ and $\vec{b}$ denote spatial vectors, not necessarily unit vectors. Confirm the beautiful identiy

$$\left(\vec{a}\cdot\hat{\vec{\sigma}}\right)\left(\vec{b}\cdot\hat{\vec{\sigma}}\right) = (\vec{a}\cdot\vec{b})\hat{1} + i(\vec{a}\times\vec{b})\cdot\hat{\vec{\sigma}}\,. \tag{10}$$

and conclude

$$\left[\vec{a}\cdot\hat{\vec{\sigma}}, \vec{b}\cdot\hat{\vec{\sigma}}\right] = 2i(\vec{a}\times\vec{b})\cdot\hat{\vec{\sigma}}\,. \tag{11}$$

(e) For unit vectors $\vec{a}, \vec{b}$ confirm

$$\langle\uparrow_a|\hat{\sigma}_b|\uparrow_a\rangle = \vec{a}\cdot\vec{b}\,, \tag{12}$$

and thus obtain the *transition probabilities*

$$p(\uparrow_b\,|\uparrow_a) \equiv |\langle\uparrow_b|\uparrow_a\rangle|^2 = \tfrac{1}{2}\left(1+\vec{a}\cdot\vec{b}\right)\,, \tag{13}$$

$$p(\downarrow_b\,|\uparrow_a) \equiv |\langle\downarrow_b|\uparrow_a\rangle|^2 = \tfrac{1}{2}\left(1-\vec{a}\cdot\vec{b}\right)\,. \tag{14}$$

(f) For many applications the ladder operators are useful. The descending operator, also called annihilation operator, is defined

$$\hat{\sigma} := \frac{1}{2}\left(\hat{\sigma}_x - i\hat{\sigma}_y\right)\,, \tag{15}$$

and thus the ascending operator, also called creation operator, is given

$$\hat{\sigma}^\dagger = \frac{1}{2}\left(\hat{\sigma}_x + i\hat{\sigma}_y\right)\,. \tag{16}$$

Confirm the matrix representation

$$\hat{\sigma} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\,, \qquad \hat{\sigma}^\dagger = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\,. \tag{17}$$

(g) Express $\hat{\sigma}_x$ and $\hat{\sigma}_y$ in terms of $\hat{\sigma}$ and $\hat{\sigma}^\dagger$, confirm the algebra

$$\hat{\sigma}^2 = \hat{\sigma}^{\dagger 2} = 0\,, \qquad \left\{\hat{\sigma}, \hat{\sigma}^\dagger\right\} = \hat{1}\,, \qquad \left[\hat{\sigma}, \hat{\sigma}^\dagger\right] = -\hat{\sigma}_z\,, \tag{18}$$

and admire the reduction in complexity: instead of using 4 different Pauli matrices, you evidently need only one (and its adjoint) in order to talk about spin-1/2 particles.

(h) Confirm the identity

$$\exp\{i\omega t\hat{\sigma}_z\}\hat{\sigma}\exp\{-i\omega t\hat{\sigma}_z\} = e^{-i\omega t}\hat{\sigma} \tag{19}$$

and alternative formulation

$$\exp\{i\omega t\hat{\sigma}^\dagger\hat{\sigma}\}\hat{\sigma}\exp\{-i\omega t\hat{\sigma}^\dagger\hat{\sigma}\} = e^{-i\omega t}\hat{\sigma} \tag{20}$$

Enjoy the formal similarity to the harmonic oscillator solution

$$\exp\{i\omega t\hat{a}^\dagger\hat{a}\}\hat{a}\exp\{-i\omega t\hat{a}^\dagger\hat{a}\} = e^{-i\omega t}\hat{a} \tag{21}$$

▷ **Aufgabe 2 (NMR)** (7 scores)

The interaction energy of a spin-1/2 magnetic moment with an magentic field is given

$$\hat{H}(t) = -\frac{\hbar}{2}\gamma\vec{B}(t)\cdot\hat{\vec{\sigma}} \tag{22}$$

where γ is the gyromagnetic ratio, and $\hat{\vec{\sigma}}$ is the Pauli spin vector (which relates $\hat{\vec{\sigma}} = \frac{2}{\hbar}\hat{\vec{s}}$ to the spin-1/2 vector $\hat{\vec{s}}$).

(a) Derive the Heisenberg equations of motion of the Pauli spin. Can you confirm the form $\dot{\hat{\vec{\sigma}}} \propto \vec{B}\times\hat{\vec{\sigma}}$? Why is that a very useful formulation? (think in terms of amount of ink, symmetry group etc).

(b) For constant magnetic field in z-direction, $\vec{B} = B_0\vec{e}_z$, and initial polarization in x-direction, $|\psi(0)\rangle = |\uparrow_x\rangle$ – what is the spin state after an interaction time T?

(c) The same as (b) but for a magnetic field which has a small rotating component $\vec{B}(t) = B_0\vec{e}_z + \vec{b}(t)$ with $\vec{b}(t) = \epsilon\cos(\omega)t\vec{e}_x + \epsilon\sin(\omega t)\vec{e}_y$.
Hint: It may be a good idea to switch to an interaction picture in which the Hamiltonian becomes time-independent. Get some inspiration from "Aufgabe 1 (h)" ....

▷ **Aufgabe 3 (Qubit Communication)** (8 scores)

Alice prepares a qubit either in state $|1\rangle = |\uparrow_a\rangle$, or in state $|2\rangle = |\downarrow_a\rangle$. The probabilities that she does the one or the other are given by $p_1 = p_2 = 0.5$.

Although Bob knows Alice's choice $\vec{a}$, he measures with orientation $\vec{b}$ (just to see how much he could learn); the two outcomes are labeled $+$ and $-$.

(a) What is the probability $P_{\sigma|i}$ that Bob finds σ, $\sigma = \pm$, given that Alice transmitted $|i\rangle$, $i = 1, 2$?

(b) What is the probability that Bob finds $\sigma = \pm$ irrespective of what Alice transmitted?

(c) What is the probability that Alice in fact transmitted $|i\rangle$ given that Bob found σ?

(d) What orientation should Bob chose for perfect communication? How many bits per qubit are revealed with this orientation?

Background This example proves that in an ideal world, where there is no disturbing influence from the enviroment, 100-percent reliable communication is possible using qubits. The only point sender and receiver must be aware of is to chose equal alignment of their Stern-Gerlach devices. The example also indicates that in this particular context, the maximal amount of information which a single qubit can carry is one bit. Later in the lecture we shall see that in a different context, which involves entanglement, one can cram two bits in a single qubit.

▷ Aufgabe 4 (Daimler-Chrysler) (12 scores)

Alice prepares a qubit in the up-state $|\uparrow_i\rangle$ with respect to one out of three possible quantization axis $\vec{a}_i$, $i = 1, 2, 3$, where the $\vec{a}_i$ form a co-planar "Mercedes-Stern",

$$\sum_{i=1}^{3} \vec{a}_i = 0\,. \tag{23}$$

Bob knows the possible directions $\vec{a}_i$, but he does not know which particular direction Alice has chosen. What is his initial level of ignorance? How much could he expect to learn about Alice's choice, and what is his optimal strategy?