Theoretische Physik V – Quanten II

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Problem set 2 Hand out: 24 Apr 2007
Hand in: 03 May 2007

Problem 2.1 – The size of a muonium atom (7 points)

A muonium atom is the bound state of a proton and a muon, a massive 'twin' of the electron (mass $\approx 100\, m_e$). For the bound state of smallest energy, the electric charge distribution is given by

$$\rho(\mathbf{r}) = e\delta(\mathbf{r}) - \frac{e\,\exp(-2r/a)}{\pi a^3} \tag{2.1}$$

where e is the electric charge quantum. The muonium Bohr radius a is significantly smaller than the conventional ('electronic') Bohr radius.

(a) Check that the muonium atom is globally neutral. (Correct the prefactors in Eq.(2.1) if needed.) Look up the ratio between muon and electron mass and find an estimate for a.

(b) An electron beam is scattered off a fixed muonium atom target, and the interaction potential $V(\mathbf{r})$ is related to the electric potential created by the charge distribution (2.1). Find its spatial Fourier transformation $\tilde{V}(\mathbf{q})$.

(c) In the lecture, it will be shown that an approximation to the cross section is given by the formula

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left|\frac{m}{2\pi\hbar^2}\tilde{V}(\mathbf{q})\right|^2 \tag{2.2}$$

where $\mathbf{q}$ is the so-called wave vector transfer: $\mathbf{q} = k\boldsymbol{\Omega} - k\mathbf{e}_z$. Discuss the cross section as a function of q. This is related to the 'form factor' of the muonium atom. For a given scattering angle, what range of electron energies $E = \hbar^2 k^2/2m$ is needed to determine with good accuracy the size of the muonium atom?

Problem 2.2 – Optical theorem (6+3* points)

Stationary scattering states are defined by the asympotic behaviour

$$\psi(\mathbf{r}) \to \mathrm{e}^{\mathrm{i}kz} + \frac{\mathrm{e}^{\mathrm{i}kr}}{r} f(\boldsymbol{\Omega}) \tag{2.3}$$

as $r = |\mathbf{r}| \to \infty$ ('in the far field'). (a) Compute the radial probability current $j_r(\mathbf{r})$ in the far field and integrate it over all possible scattering directions $\boldsymbol{\Omega}$. By taking into account the interference between the two terms in Eq.(2.3), prove the so-called 'optical theorem':

$$\frac{4\pi}{k} \mathrm{Im}\, f(\mathbf{e}_z) = \sigma \equiv \int \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \tag{2.4}$$

where $f(\mathbf{e}_z)$ is the 'forward scattering amplitude' (scattering angle $\theta = 0$). To demonstrate the optical theorem, you can use the integral

$$\lim_{r\to\infty} r \int_0^2 \mathrm{d}x \, (2 - x) \, \mathrm{e}^{\mathrm{i}rx} g(x) = 2\mathrm{i}g(0) \tag{2.5}$$

where $g(x)$ is a smooth function. (b)* Integrate by parts repeatedly to justify this formula.

Problem 2.3 – Probability density operator (4 points)

In the lecture, you have seen the Heisenberg operators for the probability density and the probability current,

$$n(q,t) = \delta(q - x(t)), \tag{2.6}$$

$$\begin{aligned} j(q,t) &= \frac{1}{2m} \{p(t),\, \delta(q - x(t))\} \\ &\equiv \frac{1}{2m} \{p(t)\, \delta(q - x(t)) + \delta(q - x(t))\, p(t)\} \, , \end{aligned} \tag{2.7}$$

where x and p are the operators of position and momentum. Demonstrate with the help of the equations of motions for these operators that the equation of continuity for the probability holds.

Problem 2.4 – Far-field propagator (3 points)

In the lecture, we needed the large-distance behaviour of the propagator for the Schrödinger equation:

$$|\mathbf{r}| \to \infty : \qquad \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \mathrm{e}^{\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{r}'} \tag{2.8}$$

where $\hat{\mathbf{r}}$ is the unit vector along $\mathbf{r}$. Prove this formula and give estimates for the terms of order $1/|\mathbf{r}|^2$ that are neglected in Eq.(2.8).