Theoretische Physik V – Quanten II

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Problem set 3 Hand out: 03 May 2007
Hand in: 08 May 2007

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30 Punkte maximal für dieses Blatt, mit einer "Scheinschwelle" von 10 Punkten wie sonst auch. Suchen Sie sich in Anbetracht der verkürzten Bearbeitungszeit die passenden Aufgaben aus.

Problem 3.1 – Lattice scattering (10 points)

In the lecture, we have used the following properties of the structure factor $S(q)$ in one dimension:

$$S(q) = \sum_{s=1}^{N} \mathrm{e}^{\mathrm{i}qsd} \tag{3.1}$$

- $S(q)$ shows peaks whenever $qd = 0 \bmod 2\pi$.
- For $N \gg 1$, these peaks have a width of order $1/(Nd)$.

Prove these properties by evaluating the sum (3.1).

In the three-dimensional case, we can construct a lattice from three linearly independent vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ that are in general not orthogonal. A finite lattice with N^3 points then leads to the structure factor

$$S(\mathbf{q}) = \sum_{s,t,u=1}^{N} \exp\left[\mathrm{i}\mathbf{q} \cdot (s\mathbf{a} + t\mathbf{b} + u\mathbf{c})\right] \tag{3.2}$$

Form a matrix A from the column vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Since these are linearly independent, the inverse of A exists. *Show* that the reciprocal lattice vectors (where $S(\mathbf{q})$ is peaked) are given by linear combinations with integer coefficients of the vectors $\mathbf{h}$, $\mathbf{k}$, $\mathbf{l}$ that form the rows of $2\pi\mathsf{A}^{-1}$. *Find* the reciprocal lattice in two dimensions for the triangular lattice: $\mathbf{a} = \mathbf{e}_x$, $\mathbf{b} = \sin\theta\,\mathbf{e}_x + \cos\theta\,\mathbf{e}_y$ with $\theta = 60°$.

Problem 3.2 – Second order Born approximation and T-matrix (10 points)

In the lecture, we have found a formula for the wave function $\psi^{(2)}(\mathbf{r})$ in the second-order Born approximation.

(a) *Compute* the corresponding scattering amplitude $f^{(2)}(\mathbf{\Omega})$.

(b) *Show* by iteration that the T-matrix also solves the equation

$$T(E) = U + T(E)G_0(E)U \tag{3.3}$$

in addition to the equation given in the lecture.

(c) *Find* by iteration the second-order approximation to the T-matrix and *show* that it gives the same scattering amplitude as in (a).

Problem 3.3 – Operator norms (10+4* points)

As you know, the operators you encounter in quantum theory are linear mappings on the Hilbert space (or suitable sub-spaces). (a) *Show* that for an operator A, the following definition defines a norm

$$\|A\| = \sup\{\|A\psi\|;\ \text{for } \|\psi\| = 1\} \tag{3.4}$$

where $\|\psi\|$ is the usual norm in the state space (the L^2-norm for wave functions).

(b) The following construction defines a linear operator similar to the one we encountered in the Lippmann-Schwinger equation. Consider a complex function of two variables $K(\mathbf{r}, \mathbf{r}')$ (also known as 'integral kernel') and be ψ a square-integrable wave function. Define the function

$$(K\psi)(\mathbf{r}) = \int \mathrm{d}^3 r'\, K(\mathbf{r}, \mathbf{r}')\psi(\mathbf{r}') \tag{3.5}$$

Identify the kernel $K(\mathbf{r}, \mathbf{r}')$ that appears in the integral term of the Lippmann-Schwinger equation. In the special case of Coulomb scattering, *is* $K\psi$ square-integrable if ψ is square-integrable?

(c) For the special case of $K(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$, *compute* the norm of K.

(d*) *Prove* the inequality

$$\|K\|^2 \leq \int \mathrm{d}^3 r\, \mathrm{d}^3 r'\, |K(\mathbf{r}, \mathbf{r}')|^2. \tag{3.6}$$