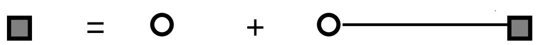
Theoretische Physik V – Quanten II

Sommersemester 2007, Universität Potsdam
Carsten Henkel

Problem set 4 Hand out: 08 May 2007
Hand in: 15 May 2007

Problem 4.1 – Feynman graphs (5 points)

The following Feynman diagram is a symbolic representation of an equation we had in the lecture. The solid line and the open circle have their usual meaning (free particle propagator, potential U)



(a) What is meant by the shaded square?

(b) Write down the iterative solution of this equation in the Feynman graph language.

(c) Write down the formula for the diagram to second order in the potential U, when taking matrix element between the free particle states $|\mathbf{k}\rangle$ and $|\mathbf{k}'\rangle$.

Problem 4.2 – Propagator and density of states (4+4* points)

The imaginary part of the propagator $G(E + \mathrm{i}\epsilon; \mathbf{x}, \mathbf{y})$ as $\epsilon \to 0$ and $\mathbf{y} \to \mathbf{x}$ plays an important role in condensed matter and electrodynamics.

(a) Show that for a particle in a box with discrete energy eigenvalues E_n,

$$\sum_n |\psi_n(\mathbf{x})|^2 \, \delta(E - E_n) = -\frac{1}{\pi} \langle \mathbf{x} | \mathrm{Im}\, G(E + \mathrm{i}\epsilon) | \mathbf{x} \rangle \tag{4.1}$$

This quantity is called the 'local density of states' (why?). Check that this equation also holds true for free particles in a very large box with periodic boundary conditions. (Answer: both sides of Eq.(4.1), in three dimensions, are proportional to $\sqrt{E}$.)

(b*) In the lecture, we have found an expression for the average propagator (its spatial Fourier transform) in a fluctuating medium of the form

$$\left(\overline{G(E; \mathbf{q})} \right)^{-1} = E - q^2 - \Sigma(E; \mathbf{q}) \tag{4.2}$$

where the so-called self-energy $\Sigma(E; \mathbf{q})$ is given by

$$\Sigma(E; \mathbf{q}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} G_0(E; \mathbf{p}) \tilde{C}[\mathbf{q} - \mathbf{p}] \tag{4.3}$$

with $\tilde{C}[\mathbf{q}]$ being the spatial Fourier transformation of the correlation function of the potential $V(\mathbf{x})$.

The quantity $\left(\overline{G(E+\mathrm{i}\epsilon;\mathbf{q})}\right)^{-1}$ is now complex, hence the poles in q of the averaged propagator have moved into the complex plane. As a consequence, $\overline{G(E+\mathrm{i}\epsilon;\mathbf{x},\mathbf{y})}$ is proportional to $\exp(-\kappa|\mathbf{x}-\mathbf{y}|)$ with $\kappa(E)>0$ even if $\epsilon\to 0$. Show that the imaginary part of Eq.(4.2), taken at $E+\mathrm{i}\epsilon$, is proportional to the total (averaged) Born scattering cross section in the fluctuating medium.

Problem 4.3 – Partial waves in one and three dimensions (5 points)

The partial wave expansion discussed in the lecture, when adapted to one spatial dimension, contains only 'even' and 'odd' wave functions as 'angular momentum quantum numbers'.

(a) Consider the reflection and transmission by a one-dimensional potential well $V(x)$ that is even with respect to the origin $x=0$. Let the potential $V(x)$ be zero for $|x|>a$. The solutions of the Schrödinger equation can be found as even or odd wave functions $\psi_{e,o}(x)$ whose behaviour for $x>a$ is

$$\psi_e(x)=\cos[kx+\delta_e(k)],\qquad \psi_o(x)=\sin[kx+\delta_o(k)] \tag{4.4}$$

Write down $\psi_{e,o}(x)$ for $x<-a$.

(b) Construct a superposition state $\psi_{\rm scat}(x)$ such that it describes an incident wave $\exp\mathrm{i}kx$ of unit amplitude coming from $x=-\infty$, including reflected and transmitted waves with amplitudes R (for $x<-a$) and T (for $x>a$). Prove that the reflection coefficient is given by

$$R(k)=\frac{\mathrm{e}^{2\mathrm{i}\delta_e(k)}-\mathrm{e}^{2\mathrm{i}\delta_o(k)}}{2}. \tag{4.5}$$

(c) Going now to three dimensions, we have discussed in the lecture a formula for the scattering amplitude $f(\mathbf{\Omega})$ where the scattering phase shifts $\delta_l(k)$ appear. Provide the missing links in the demonstration given in the lecture.

Problem 4.4 – s-wave scattering and Born approximation (6 points)

Consider the scattering of a quantum particle by a spherical square well

$$V(r)=\begin{cases}-V_0, & 0<r<a\\ 0, & a<r\end{cases} \tag{4.6}$$

(a) Write the radial Schrödinger equation for the angular momentum $l=0$ in this potential and solve it for the physically required solution. Show that the scattering phase shift is given by ($U_0=(2m/\hbar^2)V_0$)

$$\delta_0(k)=-ka+\arctan\left(\frac{k\tan(a\sqrt{U_0+k^2})}{\sqrt{U_0+k^2}}\right) \tag{4.7}$$

This phase shift gives the following contribution to the scattering amplitude, as we shall see in the lecture

$$f_{\mathrm{s}}(\mathbf{\Omega}) = \frac{\sin\delta_0(k)}{k}\mathrm{e}^{\mathrm{i}\delta_0(k)} \tag{4.8}$$

(b) In the first Born approximation for the spherical well (4.6), the scattering amplitude is given by

$$f_{\mathrm{Born}}(\mathbf{\Omega}) = \frac{U_0}{q^3}\left[qa\cos(qa) - \sin(qa)\right] \tag{4.9}$$

where, as usual, $q = 2k\sin(\theta/2)$. This expression contains contributions from all quantum numbers l. Now, it is not clear *a priori* that one can expect an agreement with Eq.(4.8). We shall analyze here the limit $ka \ll 1$ ("low energies"). Expand both Eq.(4.8) and Eq.(4.9) to third order in ka. Identify the first non-vanishing order where a difference appears. Do the validity criteria for the Born approximation (see lecture) suffice to ensure that this difference is small?