Theoretische Physik V – Quanten II

Sommersemester 2007, Universität Potsdam
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Problem set 5 Hand out: 15 May 2006
Hand in: 22 May 2006

Problem 5.1 – Bethe's formula (8 points)

In the lecture, we have seen a sketchy derivation of the so-called Bethe formula for the scattering phase shift $\delta_l(k)$ in a spherically symmetric potential $U(r)$. Provide a demonstration of the following equation

$$\sin \delta_l(k) = - \int_0^{+\infty} \mathrm{d}r \, r j_l(kr) u_l(r) U(r), \tag{5.1}$$

where $u_l(k) = r\psi_l(r)$ is the radial wave function for the angular momentum l in the potential $U(r)$ and $j_l(kr)$ the spherical Bessel function. $u_l(r)$ is normalized to the asymptotics

$$r \to \infty : \qquad u_l(r) = \sin[kr - l\pi/2 + \delta_l(k)]. \tag{5.2}$$

No guarantee for signs or factors of 2 in Eq.(5.1).

Problem 5.2 – Field theory for the Schrödinger wave function (12 points)

Recall from the mechanics lecture the Euler-Lagrange equations for a Lagrangian density $\mathcal{L}$ that depends on fields ϕ_α and their spatial derivatives $\partial_i \phi_\alpha$:

$$0 = \frac{\partial \mathcal{L}}{\partial \phi_\alpha} - \sum_i \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i \phi_\alpha)} \tag{5.3}$$

(a) Show that these equations yield the Schrödinger equation in a potential $V(\mathbf{x})$ when the following Lagrangian density is taken

$$\mathcal{L} = \frac{\mathrm{i}\hbar}{2} \left(\psi^* \partial_t \psi - (\partial_t \psi^*) \psi \right) - \frac{\hbar^2}{2m} \sum_i (\partial_i \psi^*) \partial_i \psi - \psi^* V \psi. \tag{5.4}$$

Consider as independent fields the real and imaginary parts of the complex wave function ψ.

(b) Show that the Lagrangian (5.4) changes its form when a 'local phase transformation' is applied:

$$\psi'(\mathbf{x}, t) = \mathrm{e}^{\mathrm{i}\varphi(\mathbf{x},t)} \psi(\mathbf{x}, t). \tag{5.5}$$

But if the phase $\varphi(\mathbf{x}, t)$ is constant in space and time, the Lagrangian $\mathcal{L}'$ is of the same form as Eq.(5.4).

(c) For a non-relativistic point charge q, one has to make the 'minimal coupling' replacements in the Lagrangian density

$$\partial_t \mapsto \partial_t + \frac{\mathrm{i}q}{\hbar}\phi(\mathbf{x}, t), \qquad \partial_i \mapsto \partial_i - \frac{\mathrm{i}q}{\hbar}A_i(\mathbf{x}, t), \tag{5.6}$$

where ϕ and $\mathbf{A}$ are the electromagnetic potentials. Show that in this case, the Lagrangian keeps its form when both the local phase transformation (5.5) and a gauge transformation

$$\phi' = \phi + \partial_t\chi, \qquad A_i' = A_i - \partial_i\chi \tag{5.7}$$

is applied. This is achieved for a special relation between the gauge function $\chi(\mathbf{x}, t)$ and the phase shift $\varphi(\mathbf{x}, t)$ that you are invited to identify.

This observation provides the basis for so-called gauge theories where local phase 'covariance' provides the 'minimal coupling' recipe for the interaction with the electromagnetic field. From this viewpoint, a wave function with a nontrivial phase represents a charged particle. Real wave functions correspond to neutral particles, for example, the vector 4-potential $A^\mu(x)$ and the photon.