Theoretische Physik V – Quanten II

Sommersemester 2007, Universität Potsdam
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Problem set 6

Hand out: 22 May 2006
Hand in: 29 May 2006

Problem 6.1 – Scattering of identical particles (12 points)

(a) Write down the Schrödinger equation for two particles of the same mass that interact via a potential $V(|\mathbf{r}_1 - \mathbf{r}_2|)$. Express the wave function $\psi(\mathbf{r}_1, \mathbf{r}_2)$ in terms of the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and the 'center of mass coordinate' $\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$ and work out the corresponding kinetic energy operator. Show that the dependence on the center of mass coordinate is trivial (a plane wave), while the relative coordinate corresponds to a Schrödinger equation with an 'effective mass' $\mu = m/2$. This result shows that the scattering theory for a fixed target can be translated, with minor modifications, to collisions between particles.

(b) You know from statistical physics that the two-particle wave function $\psi(\mathbf{r}_1, \mathbf{r}_2)$ for bosons (fermions) must be symmetric (anti-symmetric) when the two coordinates are exchanged:

$$\psi(\mathbf{r}_2, \mathbf{r}_1) = \pm\psi(\mathbf{r}_1, \mathbf{r}_2) \tag{6.1}$$

(For fermions, we consider here two particles in the same spin state.) This implies that the wave function $\psi(\mathbf{r})$ for the relative coordinate must be symmetric (antisymmetric) under the 'parity transformation' $\mathbf{r} \mapsto -\mathbf{r}$. Show that this implies the following 'selection' rule for the expansion of $\psi(\mathbf{r})$ into partial waves:

1. for bosons, only even partial waves $l = 0, 2, 4\ldots$ occur

2. for fermions, $l = 1, 3, 5\ldots$

In particular, 's-wave scattering' is forbidden for fermions. At low energies, fermions (in the same spin state) thus scatter much less than bosons since the scattering phase δ_1 goes faster to zero than δ_0.

Hint. Argue that without loss of generality, the relative momentum of the two particles is parallel to the z-axis. Use the symmetry properties of the spherical harmonics and the Legendre polynomials.

Problem 6.2 – Klein-Gordon propagator (8 points)

The propagator $G(\mathbf{x}, t)$ for the Klein-Gordon equation can be written as the integral ($\hbar = c = 1$)

$$G(\mathbf{x}, t) = -N \int \mathrm{d}\omega\, \mathrm{d}^3 k \frac{\mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}\cdot\mathbf{x}}}{\omega^2 - k^2 - m^2} \tag{6.2}$$

where N is a normalization constant. Show that $G(\mathbf{x}, t)$ depends only on the Minkowski distance $s^2 = t^2 - \mathbf{x}^2$. Hence, $G(\mathbf{x}, t)$ is invariant under Lorentz transformations.

Remember that $\omega t - \mathbf{k} \cdot \mathbf{x}$ is invariant if both $\mathbf{x}, t$ and $\mathbf{k}, \omega$ are changed according to the same Lorentz transformation. To proceed, you need the determinant of the transformation $(\mathbf{k}, \omega) \mapsto (\mathbf{k}', \omega')$.