Theoretische Physik V – Quanten II

Sommersemester 2007, Universität Potsdam

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Problem set 7

Hand out: 29 May 2007
Hand in: 05 June 2007

Problem 7.1 – Pauli and Dirac matrices (12 + 5* points)

(a) The Pauli matrices satisfy the relations ($i, j, k = 1, 2, 3$)

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + \mathrm{i} \epsilon_{ijk} \sigma_k \tag{7.1}$$

Starting from this property, *demonstrate* the following statements:

$$[\sigma_i,\, \sigma_j] = 2\mathrm{i}\epsilon_{ijk}\sigma_k, \qquad \mathrm{tr}\, \sigma_i = 0, \tag{7.2}$$

$$\{\sigma_i,\, \sigma_j\} := \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}, \tag{7.3}$$

$$S = \frac{1}{2} \left[\mathbb{1}\, \mathrm{tr}\, S + \sigma_i\, \mathrm{tr}\, (\sigma_i S) \right] \qquad \text{for any } 2 \times 2 \text{ matrix } S. \tag{7.4}$$

If $[S,\, \sigma_i] = 0$ for all i, then $S = c\mathbb{1}$ with a constant number c. (7.5)

(b) The basic properties of the Dirac (or gamma) matrices are ($\mu, \nu = 0, 1, 2, 3$)

- anticommutation rules
$$\{\gamma^\mu,\, \gamma^\nu\} = 2g^{\mu\nu} \tag{7.6}$$
with $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ the Minkowski metric;
- the matrices $\gamma^0 = \beta$ and $\gamma^0\gamma^i = \alpha^i$ ($i = 1, 2, 3$) are hermitean. (Why? They appear in the Dirac Hamiltonian, $\mathrm{i}\partial_t \Psi = H\Psi$.)

Starting from these properties, *show* that

$$(\gamma^0)^2 = \mathbb{1}, \qquad (\gamma^i)^2 = -\mathbb{1}, \qquad (\alpha^i)^2 = \mathbb{1}, \tag{7.7}$$

$$\mathrm{tr}\, \alpha^i = 0, \qquad \mathrm{tr}\, \gamma^\mu = 0, \tag{7.8}$$

the dimension of the matrices γ^μ is even and at least four if one is looking for four linearly independent matrices. (7.9)

(c) Consider the following set of matrices

$$J_3 = \frac{\mathrm{i}}{2} \left[\gamma^1,\, \gamma^2 \right] \tag{7.10}$$

with J_1, J_2 defined by cyclic permutations. They are hermitean. *Try to show* from (7.6) that these matrices obey the commutation relations for an angular momentum operator

$$[J_1,\ J_2] = c\mathrm{i}J_3 \tag{7.11}$$

with a real number c to be fixed and similar relations for cyclic permutations. (Bonus points.) If you do not succeed from Eq.(7.6) alone, use the explicit form of the Dirac matrices given in the lecture. (No bonus points.)

Hint. $J_3 = \mathrm{i}\gamma^1\gamma^2$ is equally true and easier to work with.

Problem 7.2 – Current density (8 points)

(a) *Prove* from the Dirac equation ($\hbar = c = 1$)

$$\mathrm{i}\partial_t\Psi = \left(m\gamma^0 - \mathrm{i}\alpha^k\partial_k\right)\Psi \tag{7.12}$$

the equation of continuity for the probability:

$$\partial_t n + \partial_k j^k = 0 \qquad \text{with} \left\{ \begin{array}{ccc} n & = & \Psi^\dagger\Psi \\ j^k & = & \Psi^\dagger\alpha^k\Psi \end{array} \right. \tag{7.13}$$

(b) In the lecture or in the textbooks, you can find the following plane wave solutions to the free Dirac equation:

$$\Psi_P = \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} \cosh(\beta/2)\psi_0 \\ \boldsymbol{\sigma}\cdot\mathbf{n}\sinh(\beta/2)\psi_0 \end{pmatrix} \tag{7.14}$$

$$\Psi_A = \begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{n}\sinh(\beta/2)\chi_0 \\ \cosh(\beta/2)\chi_0 \end{pmatrix} \tag{7.15}$$

where ψ_0 and χ_0 are normalized, but otherwise arbitrary 2-spinors. The unit vector $\mathbf{n}$ and the number β (not a Dirac matrix!) are related to the 3-momentum: $\mathbf{p} = m\mathbf{n}\sinh\beta$.

Compute the probability current $j^\mu = \overline{\Psi}\gamma^\mu\Psi = \Psi^\dagger\gamma^0\gamma^\mu\Psi$ for these states.