Theoretische Physik V – Quanten II

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Problem set 8 Hand out: 05 June 2007
Hand in: 12 June 2007

Problem 8.1 – Dirac velocity operator (4 points)

In the lecture, you have seen that in the Heisenberg picture with respect to the Dirac Hamiltonian H_D, the velocity of a Dirac particle is given by the α^k matrices ($\hbar = c = 1$)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathrm{i}\,[H_D,\, \mathbf{x}] = \boldsymbol{\alpha} \tag{8.1}$$

This is an intriguing property since the eigenvalues of $\boldsymbol{\alpha}$ are unit vectors with length 1, i.e. the particle velocity is always the speed of light.

(a) Show that $(\mathbf{n}\cdot\boldsymbol{\alpha})^2 = 1$ for a unit vector $\mathbf{n}$, using the fundamental relations for the Dirac matrices. Conclude that the eigenvalues of any component of the vector operator $\boldsymbol{\alpha}$ are ± 1.

(b) Take the Dirac representation of the α^3 matrix and construct its eigenvectors.

(c) Express the plane-wave solutions for the Dirac equation that propagate along the z-axis (see Problem 7.2), in terms of the eigenvectors found in (b) and try to resolve the paradox of the "particle moving at the speed of light".

Problem 8.2 – Non-relativistic limit (6+2* points)

This problem is an alternative calculation of the non-relativistic limit of the Dirac equation, leading to the Pauli equation.

(a) Write down the Dirac equation for a charged particle in minimal coupling to the electromagnetic field (4-vector potential $A_\mu(x)$)

$$\mathrm{i}\partial_t \Psi = H_D \Psi. \tag{8.2}$$

(b) We need the square of this equation in the special case $A_\mu = (0, -\mathbf{A})$. Write

$$(\boldsymbol{\alpha}\cdot\mathbf{D})(\boldsymbol{\alpha}\cdot\mathbf{D}) = \tfrac{1}{2}\{\boldsymbol{\alpha}\cdot\mathbf{D},\, \boldsymbol{\alpha}\cdot\mathbf{D}\} + \tfrac{1}{2}[\boldsymbol{\alpha}\cdot\mathbf{D},\, \boldsymbol{\alpha}\cdot\mathbf{D}] \tag{8.3}$$

where $\mathbf{D} = \boldsymbol{\nabla} - \mathrm{i}e\mathbf{A}$ is the covariant derivative. Now take into account the anti-commutation rules for the Dirac matrices and use the definition

$$\left[\alpha^k,\, \alpha^l\right] = 4\mathrm{i}\epsilon_{klm} S_m \tag{8.4}$$

to express the commutator in the "squared Hamiltonian" in terms of $\mathbf{S}$. In Problem 7.1, you have shown that $\mathbf{S} = \frac{1}{2}\mathbf{J}$ is an angular momentum operator. It can actually be identified with the spin of the Dirac particle. Show that $\mathbf{S}$ is a block-diagonal matrix.

(c) Go to the nonrelativistic limit by factoring the Dirac spinor into a rapidly oscillating factor $\mathrm{e}^{-\mathrm{i}mt}$ and a slowly varying spinor $\tilde{\Psi}$. Argue that this gives

$$(\mathrm{i}\partial_t)^2\Psi \approx \mathrm{e}^{-\mathrm{i}mt}(-m^2 + \mathrm{i}2m\partial_t)\tilde{\Psi} \tag{8.5}$$

Observe that the resulting equation is block-diagonal and does no longer couple "upper" and "lower" spinors ("particles" and "antiparticles"). Study the lowest-order terms in the vector potential $\mathbf{A}$ and identify the magnetic moment as the particle operator that couples linearly to the magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$.

(d)* Consider a solution with negative energy (positron) and derive the Pauli equation. What are its mass, charge, magnetic moment, and gyromagnetic factor?

Problem 8.3 – Angular momentum (4 points)

Check that the following operator

$$\mathbf{J} = -\mathrm{i}\mathbf{x} \times \boldsymbol{\nabla} + \tfrac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \tag{8.6}$$

satisfies the commutation relations of an angular momentum operator and commutes with the Dirac Hamiltonian.

Problem 8.4 – Spin algebra (6+3* points)

This is a first approach to the transformation of spinors and spinor wave functions under rotations. You know that the following quantity

$$\langle H_{SB}\rangle = g\Psi^\dagger\boldsymbol{\sigma}\cdot\mathbf{B}\Psi = g\Psi^\dagger\sigma^k\Psi B_k \tag{8.7}$$

(g is a coupling constant) gives the interaction energy of a spin with a magnetic field $\mathbf{B}$. This energy must be covariant under rotation, i.e.

$$\Psi'^\dagger\boldsymbol{\sigma}\cdot\mathbf{B}'\Psi' = \Psi^\dagger\boldsymbol{\sigma}\cdot\mathbf{B}\Psi \tag{8.8}$$

it takes the same numerical value when a rotation of coordinates leads to a different representation $\mathbf{B}'$ of the magnetic field vector. We work here with a spinor Ψ at a given position (not a wave function). It represents, for example, the spin of an atom whose position is fixed.

(a) Give an argument why the $\boldsymbol{\sigma}$ are hermitean. Give an argument why the transformation $\Psi \mapsto \Psi' = \Psi$ is not sufficient if one sticks to the conservative assumption that the set of Pauli matrices $\boldsymbol{\sigma}$ is the same in both coordinate systems.

(b) A rotation of the coordinate axes changes the components of $\mathbf{B}$ as follows: $B_i' = R_{ij}B_j$ where R_{ij} is an orthogonal 3×3 matrix. Make the *Ansatz* that the spinor transforms as $\Psi' = S(R)\Psi$ where the 2×2 matrix $S(R)$ is connected to the rotation matrix R_{ij}. *Show* that the following relation is sufficient

$$S^{\dagger}(R)\sigma_i S(R) = R_{ij}\sigma_j \tag{8.9}$$

using that $R_{ij}R_{kj} = \delta_{ik}$ (orthogonal matrix). This equation fixes $S(R)$ up to a phase factor.

(c) To find $S(R)$, consider a rotation (angle θ) around the z axis and make the ansatz

$$S(\theta) = a + b\sigma_z \tag{8.10}$$

From the properties of the Pauli matrices, *show* that one possible solution is $a = \cos(\theta/2)$, $b = \mathrm{i}\sin(\theta/2)$. (This corresponds to a particular choice of a global phase factor.) This makes S a unitary matrix. Show that in that case, the quantity $\Psi^{\dagger}\Psi$ transforms like a scalar. Give an interpretation of this quantity.

(d)* *Generalize* this result to a rotation around an arbitrary axis $\mathbf{n}$

$$S(R) = \exp\left(\tfrac{\mathrm{i}}{2}\theta\,\boldsymbol{\sigma}\cdot\mathbf{n}\right) = \cos(\theta/2) + \mathrm{i}(\boldsymbol{\sigma}\cdot\mathbf{n})\sin(\theta/2) \tag{8.11}$$

Hint. You can start from infinitesimal rotations that are of the form

$$R_{ij} = \delta_{ij} + \theta\epsilon_{ikj}n_k \tag{8.12}$$

where $\theta \ll 1$ is the rotation angle. Construct first the infinitesimal spinor transformation with the ansatz

$$S(R) = \mathbb{1} + \mathrm{i}\,\theta\,\boldsymbol{\sigma}\cdot\mathbf{a} \tag{8.13}$$

where the real vector $\mathbf{a}$ have to be fixed by Eq.(8.9). The limit to a finite (not infinitesimal) rotation angle can be performed with the identity

$$S(R) = \lim_{N\to\infty}\left(\mathbb{1} + \mathrm{i}(\theta/N)\boldsymbol{\sigma}\cdot\mathbf{a}\right)^N = \exp\left(\mathrm{i}\theta\,\boldsymbol{\sigma}\cdot\mathbf{a}\right). \tag{8.14}$$