Theoretische Physik V – Quanten II

Sommersemester 2007, Universität Potsdam
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Problem set 9 Hand out: 12 June 2007
Hand in: 19 June 2007

Problem 9.1 – Lorentz boost for a spinor (6 points)

In the lecture, we have found the following transformation rule for a Dirac spinor under a Lorentz boost along the x-axis (relative velocity $v = c\beta$ between the inertial systems:

$$\Psi'(x') = S\Psi(x), \qquad \text{with } S = \exp(\tfrac{1}{2}\beta\alpha^x) \tag{9.1}$$

(a) Derive this equation using the techniques presented in the lecture:

(i) the requirement for the transformation matrix S is that the probability current $j^\mu(x) = \overline{\Psi}(x)\gamma^\mu\Psi(x)$ transforms like a 4-vector;

(ii) for an infinitesimal Lorentz transformation, the Lorentz matrix Λ takes a simple form and the matrix S can be expanded around the unit matrix to linear order in β;

(iii) for finite values of β, exponentiate infinitesimal transformations.

(b) Check that applying this Lorentz boost, one gets the plane-wave solutions to the Dirac equation found in the lecture:

$$\Psi(x) = N \begin{pmatrix} \psi_s \\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\psi_s \end{pmatrix} \exp(-\mathrm{i}Et + \mathrm{i}\mathbf{p}\cdot\mathbf{x}) \tag{9.2}$$

(This is actually a positive-energy spinor wave function.) You can start from a spinor at rest

$$\Psi(x) = \Psi_0 \mathrm{e}^{-imt} \tag{9.3}$$

with a suitably chosen constant Ψ_0.

Problem 9.2 – The spin of the Dirac particle (3 + 3* points)

In a previous exercise, you have identified the operator

$$\mathbf{S} = \begin{pmatrix} \frac{1}{2}\boldsymbol{\sigma} & 0 \\ 0 & \frac{1}{2}\boldsymbol{\sigma} \end{pmatrix} \tag{9.4}$$

(actually a set of three operators with cartesian indices, S_1, S_2, S_3) as an angular momentum operator. You know that the total angular momentum quantum number S is related to the eigenvalue $S(S+1)$ of the "squared angular momentum" $\mathbf{S}^2 = S_1^2 + S_2^2 + S_3^2$.

Show in two ways that $S = 1/2$ by computing $\mathbf{S}^2$. First way: direct calculation with Eq.(9.4). Second way*: work with the 'abstract definitions' (see Problem 7.1.(c)) $S_1 = \frac{\mathrm{i}}{4}[\gamma^2, \gamma^3]$ (+ cyclic permutations) and use the anticommutation rules for the Dirac matrices.

Problem 9.3 – Rotation matrices (8 points)

(a) *Check* that the following definition of a linear mapping of the 3-vector $\mathbf{x}$ describes a rotation

$$R(\mathbf{n}, \theta)\mathbf{x} = \mathbf{x} + \mathbf{n} \times \mathbf{x} \sin\theta + (\mathbf{x} - \mathbf{n}(\mathbf{n} \cdot \mathbf{x}))\,(\cos\theta - 1) \tag{9.5}$$

where $\mathbf{n}$ is a unit vector. *Show* that the rotation axis is $\mathbf{n}$ and that θ is the rotation angle.

(b) Starting from Eq.(9.5), *determine* the direction of this rotation (left-handed or right-handed screw?) and *compute* the generator $J_{\mathbf{n}}$ for small θ, i.e.

$$\theta \to 0: \qquad R(\mathbf{n}, \theta)\mathbf{x} = \mathbf{x} + \mathrm{i}\theta J_{\mathbf{n}}\mathbf{x} \tag{9.6}$$

where $J_{\mathbf{n}}$ is a 3×3 matrix.

(c) Compute the commutator $[J_x,\, J_y]$ between the generators for rotations around the x and y axis and admire the connection to the angular momentum commutation relations.

Problem 9.4 – Fine structure splitting (3 points)

(a) In the lecture, the formula for the energy spectrum of hydrogen in the Dirac theory was given. Compute the splitting between the energies of the levels $2\mathrm{p}_{1/2}$ and $2\mathrm{p}_{3/2}$ to lowest non-trivial order in the fine structure constant α. Express your result in frequency units $f = |E_{1/2} - E_{3/2}|/2\pi\hbar$.

(b) Consider an electron bound to a positive point charge Ze and find an upper limit for Z so that the energy levels remain stable. What kind of nucleus would be required for that? Correct the result of (a) by including Z.