Theoretische Physik V – Quanten II

Sommersemester 2007, Universität Potsdam
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Problem set 11

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Korrigierte Fassung. Der Aufgabentext 11.1(i) in ursprünglicher Form war falsch.

Problem 11.1 – Klein-Gordon commutator (10+4* points)

The quantized neutral Klein-Gordon field can be written in the form ($\hbar = c = 1$)

$$\Phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_k V}} \left(a_{\mathbf{k}}\, \mathrm{e}^{-\mathrm{i}k_\mu x^\mu} + a_{\mathbf{k}}^\dagger\, \mathrm{e}^{\mathrm{i}k_\mu x^\mu} \right) \tag{11.1}$$

where $\omega_k^2 = k^2 + m^2$, $k_\mu = (\omega_k, -\mathbf{k})$. The operators $a_{\mathbf{k}}$, $a_{\mathbf{k}}^\dagger$ annihilate and create a Klein-Gordon particle with momentum $\mathbf{k}$. The field is enclosed in a box of volume V with periodic boundary conditions.

(i) Start from the 'classical probability density' $n(x)$ for the Klein-Gordon equation (see lecture), insert the mode expansion (11.1) and integrate over the volume V. Show that the result is either zero or the divergent quantity

$$\int \mathrm{d}V\, n(x) = -\frac{1}{2m} \sum_{\mathbf{k}} 1, \tag{11.2}$$

depending on the operator order chosen in $n(x)$.

(ii) In the quantized Klein-Gordon theory, the total number of particles is described by the operator

$$N = \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}. \tag{11.3}$$

What is the commutator between N and the annihilation/creation operators $a_{\mathbf{k}}$, $a_{\mathbf{k}}^\dagger$? Show that this result can be 'implemented' with the commutator

$$\left[a_{\mathbf{k}},\, a_{\mathbf{k}'}^\dagger \right] = \delta_{\mathbf{k},\mathbf{k}'}, \tag{11.4}$$

but also with the anti-commutation rules

$$\left\{ a_{\mathbf{k}},\, a_{\mathbf{k}'}^\dagger \right\} = \delta_{\mathbf{k},\mathbf{k}'}, \tag{11.5}$$

all other anti-commutators being zero.

(iii) Show that the commutator of the field operator with itself is given by

$$\mathrm{i}[\Phi(x),\, \Phi(0)] = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \omega_k} \sin k_\mu x^\mu \equiv D(x) \tag{11.6}$$

in the limit $V \to \infty$. Argue that the function $D(x)$ depends only on the Minkowski distance $s^2 = x_\mu x^\mu$ because of the relation (see also Problem 6.2)

$$\mathrm{d}^4 k \, \delta(k_\mu k^\mu - m^2) = \frac{\mathrm{d}^3 k}{2\omega_k} \tag{11.7}$$

Show that for any space-like separation (i.e., $s^2 < 0$), $D(x) = 0$ by analyzing $D(0, \mathbf{x})$. Show* that for time-like separations, one has

$$D(t, \mathbf{0}) = -\frac{m}{8\pi|t|} J_1(mt) \tag{11.8}$$

where J_1 is the first-order Bessel function.

Problem 11.2 – Pair production (10 points)

The one- and two-particle states for the Klein-Gordon field are given by

$$|\mathbf{p}\rangle = a^\dagger_{\mathbf{p}}|\mathrm{vac}\rangle, \qquad |\mathbf{p}, \mathbf{q}\rangle = a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}}|\mathrm{vac}\rangle \tag{11.9}$$

(i) For $\mathbf{p} \neq \mathbf{q}$, what is the difference between the states $|\mathbf{p}, \mathbf{q}\rangle$ and $|\mathbf{q}, \mathbf{p}\rangle$?

(ii) Consider an interaction operator of the form

$$V_3 = \sum_{\mathbf{p},\mathbf{q},\mathbf{k}} g_{\mathbf{pqk}} a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}} a_{\mathbf{k}} \tag{11.10}$$

and work out its matrix elements between one- and two-particle states.

(iii) The probability of decay for a Klein-Gordon particle at rest (we write $|m\rangle$ for its state) subject to the interaction (11.10) is proportional to $|\langle \mathbf{p}, \mathbf{q}|V_3|m\rangle|^2$, summed over all possible values for the momenta $\mathbf{p}, \mathbf{q}$ such that energy is conserved: $\omega_p + \omega_q = m$. Find an integral that gives this decay rate and conclude that for this example, the Klein-Gordon particle at rest is stable.