Einführung in die Quantenoptik II Sommersemester 2008

Carsten Henkel

Problem set 2

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Problem 2.1 – Photon emission (1) (5 points)

Consider a two-level atom prepared at $t = 0$ in its excited state and *compute* the rate of photon emission into a mode k, $(\mathrm{d}/\mathrm{d}t)\langle a_k^\dagger a_k \rangle$. You can use the approximations made in the lecture to derive the master equation. It may be helpful to find the "formal" solution for the atomic dipole operator $\sigma(t)$.

Show that the photon emission rate depends on time similar to Fig.2.1. The rate also depends on the mode frequency ω_k: *show* that one gets a Lorentzian lineshape centered at the (renormalized) transition frequency ω_A and a width of the order of γ.

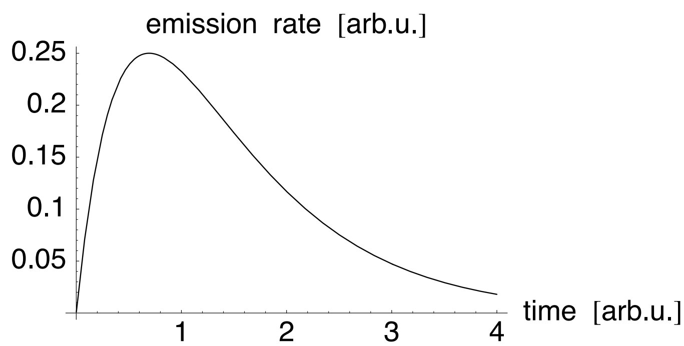

Figure 2.1: Time-dependent photon emission rate for a spontaneously decaying atom.

Problem 2.2 – Zeno effect and time-dependent decay rate (5 points)

The master equation derived in the lecture has the "Markov property", meaning that the time derivative of the atomic operators, $\mathrm{d}\boldsymbol{\sigma}/\mathrm{d}t$, depends only on the present time t. In principle, "memory effects" will occur, however, which require a "non-Markovian" description.

In the literature, one attempt towards a non-Markovian master equation is to use time-dependent expressions for the decay rate and the frequency shift:

$$\gamma(t) + \mathrm{i}\delta\omega_\mathrm{A}(t) = \int_0^t \mathrm{d}\tau\, C(\tau)\, \mathrm{e}^{\mathrm{i}\omega_\mathrm{A}\tau} \tag{2.1}$$

where $C(\tau)$ is the correlation function of the vacuum field. Consider the following "non-Markovian master equation" for the dipole operator

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\left[\gamma(t) + \mathrm{i}\omega_{\mathrm{A}} + \mathrm{i}\delta\omega_{\mathrm{A}}(t)\right]\sigma \tag{2.2}$$

and *show* that it has an exact solution where $|\langle\sigma(t)\rangle| \le |\langle\sigma(0)\rangle|$ decays in time. You may want to generalize the argument for a positive decay rate given in the lecture. *Show* that for short enough times (can you find a quantitative estimate?), the decay is quadratic (not linear) in t.

Problem 2.3 – Photon emission (2) (5 points)

The total excitation, $N = \frac{1}{2}\sigma_3 + \sum_k a_k^\dagger a_k$, is an operator that is conserved under the time evolution generated by the atom+field Hamiltonian, provided one writes the atom-field coupling in the rotating wave approximation. You have checked this fact in the previous term.

Now, consider a description in terms of a coherent state $|\alpha_{\mathrm{L}}\rangle$ for the laser field. (Note: α_{L} is time-independent.) This can be implemented by a displacement operator $D(\alpha_{\mathrm{L}})$ acting on the laser mode operator a_{L}:

$$a_{\mathrm{L}} \mapsto D(\alpha_{\mathrm{L}})^\dagger a_{\mathrm{L}} D(\alpha_{\mathrm{L}}) = a_{\mathrm{L}} + \alpha_{\mathrm{L}} \tag{2.3}$$

Compute the expression of the excitation operator after this transformation $N \mapsto N'$. Argue that N' is conserved under H'. Take the time derivative and *show* that in the stationary state, one has (the N operator here has the same form as before, but a different meaning)

$$\langle\dot{N}\rangle = -\langle\alpha_{\mathrm{L}}^* \dot{a}_{\mathrm{L}} + \mathrm{h.c.}\rangle \tag{2.4}$$

Evaluate this expression in the stationary state of the Bloch equations and discuss its dependence on the Rabi frequency.

Problem 2.4 – Negative probabilities? (5 points)

When dealing with master equations, one sometimes faces the technical problem that the density operator ρ does not remain a positive semi-definite operator at all times. Recall that this implies a violation of the inequality

$$\langle\psi|\rho(t)|\psi\rangle \ge 0 \tag{2.5}$$

for some ψ and/or some time t. This problem gives a very simple example to illustrate this issue.

Show that the density operator has a negative eigenvalue if and only if the Bloch vector $\mathbf{s}$ has a length $|\mathbf{s}| > 1$ ("one leaves the Bloch sphere"). (Hint: write the matrix elements of ρ in terms of the s_i and compute the determinant.)

Consider the master equation [recall that $s = (s_1 + \mathrm{i}s_2)/2$]

$$\begin{aligned} \dot{s}_1 &= -\delta s_2 && (2.6)\\ \dot{s}_2 &= -(\delta - \delta\omega)s_1 - \gamma s_2 && (2.7)\\ \dot{s}_3 &= -(1 + A)\Gamma s_3 + (1 - A)\Gamma && (2.8) \end{aligned}$$

where γ and Γ are positive rates, $A \geq 0$, and δ and $\delta\omega$ are real. *Comment* on the differences with respect to the Bloch equations derived in the lecture.

Show that these coefficients must satisfy the following inequalities, otherwise there are initial conditions that leave the Bloch sphere (5 bonus points; reference: Alicki and Lendi, *Quantum Dynamical Semigroups and Applications*, Springer 1987)

$$\begin{aligned} \gamma &= (1 + A)\Gamma && (2.9)\\ (1 + A)^2\Gamma^2 &\geq \gamma^2 + \delta\omega^2 + (1 - A)^2\Gamma^2 && (2.10) \end{aligned}$$

Show that both constraints are satisfied only when $A = 1$ and $\delta\omega = 0$.

Work out the stationary solution to the master equations (2.6–2.8). *Relate* A to an effective temperature of the two-level system and *comment* on the constraint $A = 1$.

Note. This master equation is an attempt to go beyond the rotating wave approximation. Most of these attempts have difficulties at some point (negative probabilities, non-thermal stationary states, violation of fundamental symmetries).