Chapter 3

Cold atom physics

In this chapter, we give an introduction to the "mechanical effects of light" on atoms, i.e., how the motion of atoms can be manipulated with laser light. This field has given rise to various techniques of laser cooling and atom trapping. It has opened the way to reach extremely low temperatures (in the nanoKelvin range) and, ultimately, the Bose-Einstein condensation of low-density gases. Recent Nobel prizes have been awarded to key scientists in this enterprise (1997: laser cooling, 2001: Bose-Einstein condensation).

3.1 Light forces

3.1.1 Electric dipole

Our working horse is still the electric dipole moment of an atom that mediates the coupling to the electric field of a laser. From the electrostatics lecture, you know that a permanent dipole in an electric field is subject to the potential energy

$$V_{\rm es} = -\mathbf{d} \cdot \mathbf{E},$$

and you get a force by taking the gradient of this quantity:

$$\mathbf{F}_{\rm es} = \sum_{i=3}^{3} d_i \nabla E_i.$$

This result is basically true also for the induced dipole of an atom or molecule. However, the calculation is a little bit more involved (i) because

the dipole is induced (should one add a gradient of $\mathbf{d}$? ... no) and (ii) the dipole is oscillating. The force thus shows oscillations at the laser frequency which are very rapid compared to the timescale on which (thermal or slow) atoms move. A more useful quantity is thus the "force averaged over one optical period". You are invited in the exercises to study this model in more detail. For a classical dipole that oscillates at the frequency of a monochromatic laser field, one gets contributions from both the electric (Coulomb) and magnetic (Lorentz) forces. Taking the average, these combine to give

$$\mathbf{F}_{\rm lf} = \sum_{i=3}^{3} d_i^* \nabla E_i(\mathbf{r}) + \text{c.c.}$$

where now $\mathbf{d}$ and $\mathbf{E}$ are the positive frequency amplitudes of the dipole and field. (We recall our convention that $\mathbf{E}(t) = \mathbf{E}\,\mathrm{e}^{-\mathrm{i}\omega_L t} + \mathbf{E}^*\,\mathrm{e}^{\mathrm{i}\omega_L t}$.)

In the exercises, you are invited to analyze the light force for a classical electric dipole. Assuming that the dipole responds linearly to the field (with a polarizability $\alpha(\omega_L)$ dependent on the laser frequency), you will find contributions proportional to the absorption ($\mathrm{Im}\,\alpha$) and the dispersion ($\mathrm{Re}\,\alpha$). A similar distinction can be made for the more quantum-mechanical two-level atom that we discuss now.

3.1.2 Two-level atom

The dipole now becomes an operator (we still treat the light field classically), and we have to include an additional average over the quantum state of the two-level atom. This state will in general be a mixed one, and the corresponding density matrix element is (we need the positive frequency component of the dipole operator)

$$\langle \mathbf{d} \rangle = \mathbf{d}_{\rm eg} \langle \sigma \rangle$$

where $\mathbf{d}_{\rm eg}$ is the c-vector of dipole matrix elements and $\langle \sigma \rangle = \rho_{\rm eg}$ is the optical coherence in the rotating frame. The average light force is now given by

$$\langle \mathbf{F}_{\rm lf} \rangle = \sum_i \left(\rho_{\rm eg}^* D_i^* \nabla E_i(\mathbf{r}) + \text{c.c.} \right) = -\hbar\, \mathrm{Re}\left(\rho_{\rm eg}^* \nabla \Omega(\mathbf{r}) \right) \qquad (3.1)$$

where in the last step we have introduced the Rabi frequency (and the c-vector $\mathbf{D}$, being constant, has been included under the gradient).

When a laser field of a typical duration (longer than the natural lifetime $1/\gamma$ of the excited state) is applied on the two-level atom, the atomic density matrix can safely be evaluated in the steady state. We recall that in this limit, the atomic coherence and the excited state population are given by

$$\rho_{\rm eg} = \frac{\Omega}{2}\frac{\Delta - {\rm i}\gamma}{\Delta^2+\gamma^2+|\Omega|^2/2} \tag{3.2}$$

$$\rho_{\rm ee} = \frac{|\Omega|^2/4}{\Delta^2+\gamma^2+|\Omega|^2/2} \tag{3.3}$$

where Δ is the detuning of the laser field and 2γ the decay rate of the excited state. In the following, we discuss two contributions to the force (3.1) related to imaginary and real part of this optical coherence. We see here that the steady state $\rho_{\rm eg}$ is position-dependent. Since the force (3.1) only involves the gradient of the Rabi frequency, it is clear that in general the force cannot be written as the gradient of a potential.

3.1.3 Radiation pressure force

We can always write the gradient of the Rabi frequency in (3.1) in terms of an amplitude and a phase gradient:

$$\nabla\Omega = \frac{\Omega}{|\Omega|}\nabla|\Omega| + {\rm i}\nabla\phi\,\Omega$$

where ϕ is the phase of Ω. These two gradients give two contributions to the light force:

$$\langle\mathbf{F}\rangle = \mathbf{F}_{\rm rp} + \mathbf{F}_{\rm dip} \tag{3.4}$$

$$\mathbf{F}_{\rm rp} = \hbar 2\gamma\,\nabla\phi\frac{|\Omega|^2/4}{\Delta^2+\gamma^2+|\Omega|^2/2} \tag{3.5}$$

$$\mathbf{F}_{\rm dip} = -\frac{\hbar\Delta}{4}\frac{\nabla|\Omega|^2}{\Delta^2+\gamma^2+|\Omega|^2/2} \tag{3.6}$$

To give an interpretation of these two contributions, let us first consider a plane wave laser field. The Rabi frequency then has constant modulus, and only a phase gradient occurs. The light force is then given by

$$\mathbf{F}_{\rm rp} = \hbar\mathbf{k}\,2\gamma\frac{|\Omega|^2/4}{\Delta^2+\gamma^2+|\Omega|^2/2} = \hbar\mathbf{k}\,2\gamma\,\rho_{\rm ee}. \tag{3.7}$$

We see that the force is oriented parallel to the wave vector of the laser field or parallel to the "photon momentum" $\hbar\mathbf{k}$. The atom is thus "pushed forward" by the laser beam — this force is called "radiation pressure". With the photon picture, we can give an intuitive interpretation of the radiation pressure: it is given by the photon momentum times the number of times per unit time, $2\gamma\rho_{\mathrm{ee}}$ that the atom decays spontaneously from its excited state. This number occurs because each spontaneous emission has been preceded by the absorption of a photon from the laser field, giving the atom a momentum $\hbar\mathbf{k}$. If the photon emission is stimulated, back into the laser beam, the atom also gives back the absorbed momentum. But if the emission is spontaneous, the photon is emitted in a random direction (with a probability given by the dipole radiation pattern), and there is no net momentum transfer on average.

Let us make an order of magnitude estimate for the radiation pressure force. The corresponding acceleration is maximally

$$a_{\mathrm{rp}} \leq \frac{\hbar k}{2m} 2\gamma \sim \frac{10^{-34}\,\mathrm{J\,s}}{10^{-26}\,\mathrm{kg}} \frac{10^{7}\,\mathrm{s}^{-1}}{10^{-7}\,\mathrm{m}} \sim 10^{8}\,\mathrm{m/s}^{2}$$

— this is really large: ten millions times the gravitational acceleration.

With this force, one can decelerate an atom very fast: if we start with an atomic beam at a typical thermal velocity (for a few hundred K), $v \sim 10^{3}\,\mathrm{m/s}$, then the beam should come to rest on a timescale

$$\frac{v}{a_{\mathrm{rp}}} \sim \frac{10^{3}\,\mathrm{m/s}}{10^{8}\,\mathrm{m/s}^{2}} = 10^{-5}\,\mathrm{s},$$

which corresponds to a distance of about $v^2/a_{\mathrm{rp}} \sim 1\,\mathrm{cm}$. But this estimate neglects the Doppler effect: one has to adjust the laser frequency to stay in resonance with the atomic transition. If this is not done, the atom gets out of resonance after a velocity change Δv such that the Doppler shift $k\Delta v \sim \gamma$. This gives

$$\Delta v \sim \frac{\gamma}{k} \sim \frac{10^{7}\,\mathrm{s}^{-1}}{10^{7}\,\mathrm{m}} = 1\,\mathrm{m/s}$$

which is not very much compared to a thermal velocity. In practice, there are two options to keep in resonance: either the laser frequency is "chirped"[1]and is getting smaller while the atoms slow down (smaller

[1]to chirp: *zwitschern*.

Doppler shift). Or one uses the Zeeman effect in a magnetic field that varies in space. The atomic beam flows through a tube where the field is initially strong to compensate the large Doppler shift and gets gradually weaker when the atoms get slower.

The Doppler effect also plays a crucial role to cool atoms. This we shall see in the next section.

3.1.4 Dipole force

What about the second contribution $\mathbf{F}_{\rm dip}$ in (3.6)? It requires an intensity gradient, and this may remind us of the shift of the two atomic "dressed" levels with respect to the "bare" transition frequency. Let us focus to simplify things on the limit of a large detuning, $\Delta \gg |\Omega|, \gamma$.

In that limit, we can neglect spontaneous emission and use the formulas of the previous term for the dressed levels: $\Delta E_{g,e} = \pm\frac{1}{2}\hbar\Delta\sqrt{1+|\Omega|^2/\Delta^2}$ for the ground and excited states. Taking the negative gradient, these energy shifts give a force

$$\mathbf{F}_{\rm g,e} = \mp\frac{\hbar}{4\Delta}\nabla|\Omega|^2$$

Assuming now that the atom, moving not too fast in the laser field, follows the dressed level connected to the ground state, we get the large-detuning limit of (3.6). The dipole force can thus be understood from the spatially varying eigenenergies of the atom-field interaction Hamiltonian.

A more careful analysis shows that even for $\Delta \sim \Omega \sim \gamma$, one may still use the idea of a gradient of level shifts, averaged over the atomic populations:

$$\langle \mathbf{F} \rangle = -\sum_{\alpha=1,2} \nabla E_\alpha \rho_{\alpha\alpha} \tag{3.8}$$

But the transitions that occur between these levels due to spontaneous emission have to be calculated more carefully, since both are a mixture of ground and excited states.

To conclude, let us mention that the dipole force derives from a potential

$$\begin{aligned} U_{\rm dip}(\mathbf{r}) &= \frac{\hbar\Delta}{2}\log\Big(\Delta^2+\gamma^2+|\Omega(\mathbf{r})|^2/2\Big) \\ &= \frac{\hbar\Delta}{2}\log\left(1+\frac{\gamma^2}{\Delta^2+\gamma^2}\frac{I(\mathbf{r})}{I_{\rm sat}}\right) + \text{const.} \end{aligned}$$

In the second line, we have used the "saturation intensity" that is characteristic for the atomic transition used (it is related to the linewidth γ and the transition wavelength). It often happens that one works in the limit of large detuning (to enhance the dipole force relative to radiation pressure). In this regime, the dipole potential can be written

$$U_{\rm dip}(\mathbf{r}) \approx \frac{\hbar|\Omega(\mathbf{r})|^2}{4\Delta} = \frac{\hbar\gamma^2}{2\Delta}\frac{I(\mathbf{r})}{I_{\rm sat}}$$

which is proportional to the light intensity.

Note that the dipole potential is attractive towards the regions of large intensity when the detuning is negative ("red detuning"). In this way, atoms can be trapped in the focus of a far-detuned laser beam ("optical tweezer"). Another simple dipole trap are the nodes of a standing laser wave with "blue" (= positive) detuning. This trap has the advantage that at the bottom of the potential well, the atoms are "in the dark" (the light intensity is zero at the nodes), and spontaneous emission is suppressed. Finally, with an "evanescent" wave that shows an intensity gradient perpendicular to a glass surface (on its back side, a laser beam is totally reflected), one can realize one-dimensional attractive or repulsive potentials — the repulsive potential being an interesting candidate for an "atom mirror" if the atoms arrive with a not too large kinetic energy.

3.2 Doppler cooling

This is the code word for a simple cooling mechanism that works with two-level atoms. It has been discovered by Hänsch and Schawlow in 1975.

We have seen that the radiation pressure force can exert a very large acceleration on an atom provided the laser field is kept on resonance. This can be used to slow down atoms, but it is not yet cooling.

3.2.1 Light friction: "optical molasses"

Cooling means that we also want to reduce the velocity spread of an atomic ensemble. This can be done, at least in one dimension, in a configuration with two laser beams, impinging from the left and right onto the atom.

We have to take into account the Doppler shift of the laser frequency when the atom is moving. In a single plane wave beam with wave vector $\mathbf{k}$, the Doppler shift leads to a velocity-dependent detuning:

$$\omega_L \mapsto \omega_L - \mathbf{k}\cdot\mathbf{v} \Longrightarrow \Delta \mapsto \Delta - \mathbf{k}\cdot\mathbf{v}$$

The laser frequency increases in the frame of the moving atom when atomic velocity and laser wave vector are opposite — remember the tune of the fire department car.

We thus get a velocity-dependent radiation pressure force

$$\mathbf{F}_{\rm rp}(\mathbf{v}) = \hbar\mathbf{k}\,2\gamma\frac{\Omega^2/4}{(\Delta-\mathbf{k}\cdot\mathbf{v})^2+\gamma^2+|\Omega|^2/2}.$$

In the following, we put $\gamma'^2 = \gamma^2 + |\Omega|^2/2$.

If we have two counter-propagating beams with wave vectors $\pm\mathbf{k}$ and the same power and if we are allowed to sum their radiation pressure forces, we get the following total force (see Fig.3.1)

$$\mathbf{F}_{\rm Dc}(\mathbf{v}) = \frac{\hbar\mathbf{k}\,2\gamma\Omega^2}{4}\left(\frac{1}{(\Delta-\mathbf{k}\cdot\mathbf{v})^2+\gamma'^2} - \frac{1}{(\Delta+\mathbf{k}\cdot\mathbf{v}^2+\gamma'^2}\right). \qquad (3.9)$$

This function shows a positive peak for velocities with $\mathbf{k}\cdot\mathbf{v} = \Delta$ (radiation

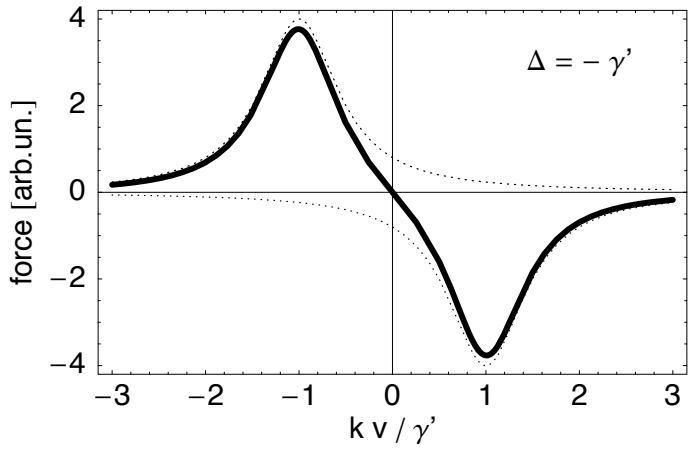

Figure 3.1: Radiation pressure force (3.9) as a function of atomic velocity $\mathbf{k}\cdot\mathbf{v}$. The force is normalized to $\hbar k 2\gamma(\Omega/2\gamma')^2$. Note that in this figure, $\gamma' = 2(\gamma^2+|\Omega|^2/2)^{1/2}$.

pressure of the beam coming from the left) and a negative peak for $\mathbf{k}\cdot\mathbf{v} = -\Delta$ (beam from the right). If we choose a detuning $\Delta < 0$, we get a force that is opposite to the direction of motion in the interval $\Delta < \mathbf{k}\cdot\mathbf{v} < -\Delta$. The atom moves to the right and the beam coming from the right gets closer to resonance, giving a larger radiation pressure. The atom moves to the left, and the beam from the left comes into resonance. This situation is called an "optical molasses"[2] because the atomic velocity gets rapidly damped to zero once it lies in the interval mentioned above.

In particular, around zero velocity we get

$$\mathbf{F}_{\mathrm{Dc}}(\mathbf{v}) \approx \hbar\mathbf{k}\,\Omega^2(\mathbf{k}\cdot\mathbf{v})\frac{2\Delta\gamma}{(\Delta^2+\gamma'^2)^2}.$$

In a one-dimensional problem, we thus have a "friction force"

$$\begin{aligned} F_{\mathrm{Dc}} &= -m\alpha v \\ \text{with } \alpha &= -\frac{\hbar k^2}{m}\Omega^2\frac{2\Delta\gamma}{(\Delta^2+\gamma'^2)^2} \end{aligned} \tag{3.10}$$

This friction allows to cool the atom: the atomic kinetic energy decays like

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{mv^2}{2} = -m\alpha v^2.$$

The maximum value of the friction coefficient is obtained for a detuning $\Delta \sim -\gamma$ (up to a numerical factor that you are invited to compute in the exercises).

3.2.2 The Doppler limit temperature

The equilibrium temperature in optical molasses ("Doppler limit") is determined by the balance between friction (reduces the kinetic energy) and the increase of the atomic momentum due to the random recoil momenta occurring in spontaneous emission. These random recoils result in a random walk of the atomic momentum and a linear increase of the momentum variance as times goes on:

$$\langle \delta p^2(t)\rangle = 2D_p t$$

[2]molasses: *Sirup, Abfallprodukt bei der Herstellung von Rum*

Here, D_p is called the momentum diffusion coefficient. If the average momentum of the atom is zero, we may also write this equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle p^2\rangle\bigg|_{\mathrm{diff}} = 2D_p$$

The balance between friction and diffusion is therefore given by

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\langle p^2\rangle}{2m} = -\frac{\alpha}{m}\langle p^2\rangle + \frac{D_p}{m}$$

and we get an equilibrium kinetic energy ('temperature')

$$\frac{1}{2}k_B T = \frac{\langle p^2\rangle}{2m} = \frac{D_p}{2m\alpha} \tag{3.11}$$

The larger the friction coefficient, the lower the temperature. This is an example of an *Einstein relation* or *fluctuation–dissipation theorem* that links dissipation (here: friction) to fluctuation (here: momentum diffusion). It was first introduced in 1905 in Einstein's paper on Brownian motion.. In the case of laser cooling, the Einstein relation is used to actually define the limiting temperature. Note that there is no guarantee that the actual momentum distribution of the atoms is of Maxwell-Boltzmann form (in reality, it is not).

A simple model for the momentum diffusion is based on the random photon momenta that the atom exchanges with the spontaneously emitted photons. The emission processes occur with a rate $\Gamma = 2\gamma\,\rho_{\mathrm{ee}}$ such that $N = \Gamma\,\delta t$ emissions occur during a time δt.

The variance of the photon recoil momenta is (more precisely, we take the projection onto the direction of the cooling laser)

$$\begin{aligned}\langle \delta p^2\rangle &= \Big\langle \Big(\sum_{\alpha=1}^{N}\hbar k_{x,\alpha}\Big)^2\Big\rangle = N\hbar^2\left\langle k_x^2\right\rangle = \dots \\ &= 2\gamma\delta t\,\rho_{\mathrm{ee}}\frac{2}{5}\hbar^2k^2\end{aligned}$$

We use the fact that each spontaneous photon is emitted independently of the others and for each one, the same distribution applies. In the last step, we have used the dipole radiation pattern as the probability distribution for the directions of the emitted photons. The dipole moment is assumed to be

linearly polarized and orthogonal to the cooling laser (accurate for a linear laser polarization). Finally, the magnitude k of the spontaneous photon wavevector is nearly equal to that of the cooling laser (the difference is of the order of Δ/c, but this is very small compared to $k = \omega/c$).

We thus see that the momentum variance increases linearly with time δt and get a diffusion coefficient

$$D_p = \frac{\delta p^2}{\delta t} = \frac{2}{5}\gamma\hbar^2 k^2 \frac{\Omega^2/4}{\Delta^2 + \gamma'^2}$$

where we have used for $\rho_{\rm ee}$ the steady state value for an atom with $v = 0$.

Finally, the lowest temperature is given by (may be wrong by a factor 2)

$$\begin{aligned} k_B T_D &= \min \frac{D_p}{m\alpha} = \ldots = \min \frac{-\hbar}{20} \frac{\Delta^2 + \gamma'^2}{\Delta} \\ &= \frac{\hbar}{20} \min \sqrt{4\gamma^2 + 2\Omega^2} = \frac{\hbar\gamma}{10} \end{aligned} \tag{3.12}$$

As an order of magnitude, $T_D \approx 10\,\mu$K for typical linewidths.

3.3 Atom diffraction by light

Simplest model with Hamiltonian

$$H = \frac{p^2}{2M} - \frac{\hbar\Delta}{2}\sigma_3 + g\left(\sigma^\dagger a\, {\rm e}^{{\rm i}kx} + a^\dagger \sigma\, {\rm e}^{-{\rm i}kx}\right) \tag{3.13}$$

with spatial mode function proportional to $g\,{\rm e}^{{\rm i}kx}$. Momentum operator p is conjugate to position x. Typical setting: atomic beam crosses standing wave, motion along the beam is treated classically (fixed momentum Mv_z, interaction time t set by v_z and width of standing wave along z, not explicitly written down here).

Solution on resonance and in the large-detuning limit: see exercise and the book by Orszag.

3.4 Adiabatic passage and slow light

In this section, we discuss two effects that have been recently developed in great detail with applications for quantum information processing. The

main application could be a “memory” for a “photonic qubit”. The idea is to slow down a light pulse and even to stop it in an atomic medium. More generally, the techniques that are used here (“adiabatic passage”) can be employed to manipulate with laser beams information that is encoded in stable ground states of atoms.

3.4.1 Adiabatic passage

This method, called “adiabatic passage” or “STIRAP” (probably “stimulated Raman transitions for adiabatic passage”) has the following advantages:

- The atomic levels a manipulated without directly populating the excited state. If this would happen, one would have a nonzero rate of spontaneously emitted photons. This rate limits the accuracy of the operation, in a way similar to the effect of dissipation in the Rabi flopping of a two-level atom.

The idea is to identify a superposition of ground state sublevels that is *not coupled* to the laser field. At first sight, it seems strange that this strategy can succeed – but this is precisely what we shall see.

Simplified level scheme

The adiabatic passage idea can be implemented in atoms that have several magnetic sublevels in the ground state. This happens when the electron or nuclear spin are taken into account, or when the ground state is not an s-state ($l = 0$). We use here the so-called Λ-scheme shown in figure 3.2.

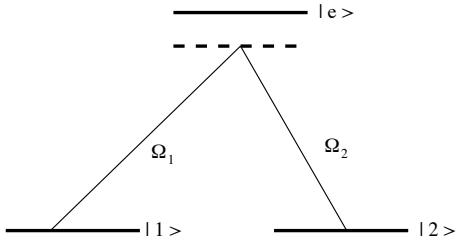

Figure 3.2: Three level scheme used for adiabatic passage.

We first write down the Hamiltonian for the atomic levels and the coupling to the lasers 1 and 2 in the basis $|\mathrm{e}\rangle\,,\,|1\rangle\,,\,|2\rangle$. We make the rotating

wave approximation and choose a rotating frame that removes the time-dependence of the laser coupling. This gives in matrix form

$$H = \hbar \begin{pmatrix} -\Delta & \frac{1}{2}\Omega_1 & \frac{1}{2}\Omega_2 \\ \frac{1}{2}\Omega_1^* & 0 & 0 \\ \frac{1}{2}\Omega_2^* & 0 & 0 \end{pmatrix} \tag{3.14}$$

Eigenvalues and dark state

The eigenvalues of this Hamiltonian are simple to find. The characteristic polynomial gives

$$0 = \det \begin{pmatrix} -\Delta - \lambda & \frac{1}{2}\Omega_1 & \frac{1}{2}\Omega_2 \\ \frac{1}{2}\Omega_1^* & -\lambda & 0 \\ \frac{1}{2}\Omega_2^* & -\lambda & 0 \end{pmatrix} = -(\lambda + \Delta)\lambda^2 + \frac{\lambda}{4}\left(|\Omega_1|^2 + |\Omega_2|^2\right).$$

We thus find the following eigenvalues

$$\lambda_a = 0, \qquad \lambda_{s,e} = -\frac{\Delta}{2} \pm \frac{1}{2}\sqrt{\Delta^2 + |\Omega_1|^2 + |\Omega_2|^2}$$

where it is interesting that one eigenvalue is zero. If we plot these eigenvalues for "pulsed lasers" where the total Rabi frequency $\Omega = \left(|\Omega_1|^2 + |\Omega_2|^2\right)^{1/2}$ varies with time in some gaussian manner, we get the energy levels shown in figure 3.3. We see that two levels "repel" each other with an energy dif-

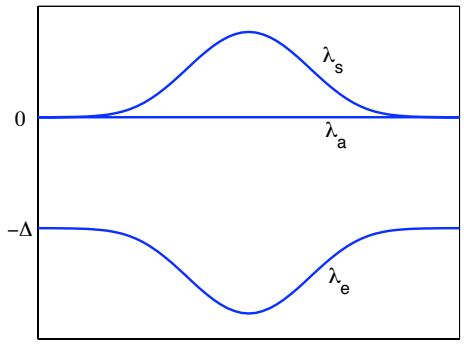

Figure 3.3: Instantaneous eigenvalues of the three level Hamiltonian for a total Rabi frequency $|\Omega_1|^2 + |\Omega_2|^2$ with gaussian time-dependence.

ference given by $(\Delta^2 + |\Omega_1|^2 + |\Omega_2|^2)^{1/2}$ similar to what we know from the "dressed states" of a two-level atom. But the third level remains at energy zero and behaves as if the laser fields were not there — for this reason, the corresponding state is called "dark".

The dark state is easy to find

$$|a\rangle = \frac{1}{\Omega}\left(\Omega_2 |1\rangle - \Omega_1 |2\rangle\right)$$

and is given by an "antisymmetric" superposition of the ground states $|1\rangle, |2\rangle$. (For this reason, we have labelled it "a".) Sometimes people insist that this state is dark because of "quantum interference": its superposition of ground state probability amplitudes is made in such a way that the amplitudes interfere destructively with respect to the laser coupling. Indeed, by construction the Hamiltonian (3.14) gives zero when acting on $|a\rangle$. In the exercises, you are asked to compute the other, "bright" eigenstates.

Adiabatic pulse sequence

We now show how with a sequence of laser pulses, one can transform the ground states $|1\rangle, |2\rangle$ into each other fast and without passing via the excited state (that is subject to spontaneous decay).

The *idea* is to make the Rabi frequencies slowly time-dependent so that the atom stays in one of the eigenstates determined by the instantaneous Rabi frequencies. This limit is called the "adiabatic limit". In the exercises, you are asked to work out a condition on the time-dependences of the Rabi frequencies that ensures adiabaticity. As a general rule, the characteristic timescale τ for the time-dependence of $\Omega_{1,2}(t)$ has to be large compared to the inverse frequency differences between the instantaneous eigenstates. Note that this condition cannot be satisfied and has to be refined when a degeneracy occurs in the eigenenergies (as in our case for $\Omega_{1,2} \to 0$).

The dark state is an ideal candidate as a "transit state" to perform the transformation $|1\rangle \to |2\rangle$. We note that it approaches the "target" ground state $|2\rangle$ when $\Omega_2 \to 0$ (up to a phase) and similarly $|a\rangle \to |1\rangle$ when $\Omega_1 \to 0$. Schematically,

$$|1\rangle \overset{\Omega_1 \to 0}{\longleftarrow} |a\rangle \overset{\Omega_2 \to 0}{\longrightarrow} |2\rangle .$$

The pulse sequence is thus the following (see figure 3.4): first switch on Ω_2 to identify the state $|1\rangle$ with the dark state $|a\rangle$. Then switch on Ω_1 to change

the relative weights of the ground states in the dark state. Shut down Ω_2 before Ω_1, this identifies the dark state and the target state $|1\rangle$.

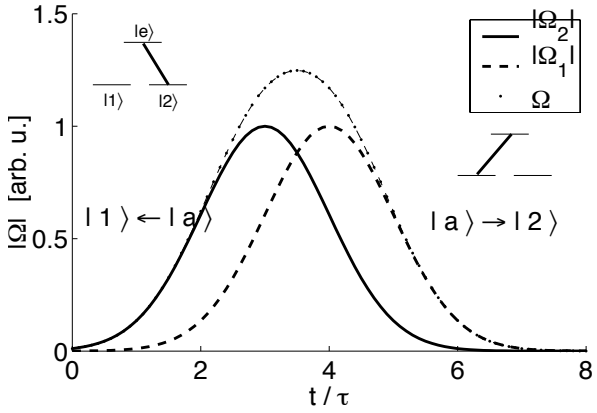

Figure 3.4: Counterintuitive pulse sequence for adiabatic passage.

Note that this pulse sequence is "counterintuitive" because one first drives the transition $e \leftrightarrow 2$ where is no population. If we made the transformation via the excited state, one would first drive the transition $e \leftrightarrow 1$ to couple to the (initial) population in state $|1\rangle$. But the counterintuitive sequence is adapted to work with the dark state. Indeed, the initial state $|1\rangle$ is dark to the laser pulse on the transition $e \leftrightarrow 2$. Similarly, at the end of the sequence, the laser on the transition $e \leftrightarrow 1$ may be left on, while the atom is in the "dark" state $|2\rangle$.

Let us recall the advantages of this "stimulated Raman adiabatic passage":

- the excited state is not involved, and the scheme is robust with respect to spontaneous decay;
- the operation can be made "fast" (its timescale is given by the duration of the pulses) and is only limited by the requirement of being adiabatic. This is ensured with sufficiently strong laser fields and/or some finite detuning and results in a faster operation than with a stimulated Raman transition.

For these reasons, STIRAP is generally regarded as a good candidate to flip a qubit in quantum computing.

3.4.2 Slow light or: "electromagnetically induced transparency"

We now focus on the behaviour of the light field when a dark state is present in an atom. We can expect that something special occurs in the dark state, since the atom "does not see the light". Indeed, we shall see that when the laser Ω_2 is on, the atom becomes "transparent" to the laser Ω_1 – no absorption nor dispersion occurs. This is called "electromagnetically induced transparency". In addition, this transparency has a narrow width in frequency such that when a laser pulse on the transition $e \leftrightarrow 1$ is sent through the atom(ic medium), its group velocity can be dramatically reduced. It has been shown in a recent experiment that group velocities of the order of a few metres per second are possible and that the light pulse can even be "stopped" inside the medium by shutting off the laser Ω_2. (Switching it on later restores the pulse – one can thus potentially store an optical qubit for a "long time".)

In this lecture, we sketch the main ingredients of this phenomenon. We recall the concept of group velocity and refractive index. Then the group velocity is computed using the Bloch equations for a three level medium.

Group velocity and index

We recall that the dispersion relation for light in a (lossless) medium with an index $n(\omega)$ is given by $k = \omega n(\omega)/c$. For a narrow-frequency pulse, the group velocity is defined by

$$\frac{1}{v_g} = \frac{\mathrm{d}k}{\mathrm{d}\omega} = \frac{1}{c}\left(n + \omega\frac{\mathrm{d}n}{\mathrm{d}\omega}\right). \tag{3.15}$$

The refractive index of the medium is related to the induced polarization, as we learnt in the chapter on laser theory. In our case, with two lasers, the laser polarization selects the relevant levels that are coupled. We focus in the following on the dispersion and absorption of the laser Ω_1. In terms

of the density matrix element ρ_{e1} for the relevant transition $e \leftrightarrow 1$, the dispersion (real part of the refractive index) is given by

$$n - 1 = -\frac{D_{e1}^2 N}{\hbar \varepsilon_0} \mathrm{Re} \frac{\Omega_1^* \rho_{\mathrm{e1}}}{|\Omega_1|^2} \tag{3.16}$$

where we have expressed the electric field in terms of the Rabi frequency, and D_{e1} is the dipole matrix element and N the atomic density.

The (inverse) group velocity is thus related to the slope of the optical coherence with respect to the frequency of laser 1:

$$\frac{\mathrm{d}\rho_{\mathrm{e1}}}{\mathrm{d}\omega_1} = \frac{\mathrm{d}\rho_{\mathrm{e1}}}{\mathrm{d}\Delta_1}.$$

In the second step, we have introduced the detuning $\Delta_1 = \omega_1 - \omega_{e1}$.

In the experimental paper of 1999 [*Nature* **397**, 594], Lene V. Hau & *al.* quote the following expression for the inverse group velocity (in our notation):

$$\frac{c}{v_g} \approx \frac{2\omega_1 D_{e1}^2 N}{\hbar \varepsilon_0 |\Omega_2|^2}. \tag{3.17}$$

Using the formula for the linewidth γ of the transition $e \leftrightarrow 1$, $\gamma = D_{e1}^2 (2\pi/\lambda_{e1})^3 / 3\pi\hbar\varepsilon_0$, we see that the group velocity can indeed be made very small with a sufficient density

$$\frac{c}{v_g} \sim (N\lambda_{e1}^3) \frac{\omega_{e1}\gamma}{|\Omega_2|^2}$$

because typically the Rabi frequency Ω_2 is much smaller than the transition frequency ω_{e1}. The prefactor is unity for a density of the order of $10^{15}\,\mathrm{cm}^{-3}$ which is typical for a Bose-Einstein condensate. (The effect is less dramatic in a thermal gas because the coherence between the ground states $|1,2\rangle$ dephases due to collisions.)

We shall see in the following how this result for the group velocity comes about.

Bloch equations

Since we now want to vary the detuning ω_1 of the "probe laser", we have to modify the model used in the previous subsection. In the exercises, you

show that in a suitable rotating frame, the three-level atom is described by the Hamiltonian

$$\begin{aligned} H &= \hbar\Delta_1 |1\rangle\langle 1| + \hbar\Delta_2 |2\rangle\langle 2| \\ &\quad + \frac{\hbar}{2}\left(\Omega_1 |e\rangle\langle 1| + \text{h.c.}\right) + \frac{\hbar}{2}\left(\Omega_2 |e\rangle\langle 2| + \text{h.c.}\right). \end{aligned} \tag{3.18}$$

We assume in the following that the laser Ω_2 (the "coupling laser") is tuned to resonance so that $\Delta_2 = 0$.

We now need the relaxation part of the Liouville equation for the three level atom. The easiest way is to use an effective Hamiltonian and the "jump part" of the Liouvillian. We assume that the excited state $|e\rangle$ decays with a branching ratio $50 : 50$ into the two ground states. The effective Hamiltonian is thus given by

$$H_{\text{eff}} = H - \mathrm{i}\hbar\gamma |e\rangle\langle e| \tag{3.19}$$

where 2γ is the total decay rate of $|e\rangle$. The imaginary part of this Hamiltonian gives decay terms in the rate equations for ρ_{ee} and the decay of the optical coherences $\rho_{e1},\ \rho_{e2}$. The jump part that describes how the population arrives in the lower levels is given by a sum of two Lindblad operators

$$\sum_{k=1,2} L_l \rho L_k^\dagger = \gamma |1\rangle\langle 1| \rho_{ee} + \gamma |2\rangle\langle 2| \rho_{ee}. \tag{3.20}$$

The optical Bloch equations for the three level atom break into six lines for the relevant density matrix elements. Using the Hamiltonian (3.18) and the relaxation parts, we find

$$\dot{\rho}_{ee} = -2\gamma\,\rho_{ee} - \frac{\mathrm{i}}{2}\left(\Omega_1\rho_{1e} - \Omega_1^*\rho_{e1}\right) - \frac{\mathrm{i}}{2}\left(\Omega_2\rho_{2e} - \Omega_2^*\rho_{e2}\right) \tag{3.21}$$

$$\dot{\rho}_{e1} = \left(-\gamma + \mathrm{i}\Delta_1\right)\rho_{e1} - \frac{\mathrm{i}}{2}\left(\Omega_1\rho_{11} - \Omega_1\rho_{ee}\right) - \frac{\mathrm{i}}{2}\Omega_2\rho_{21} \tag{3.22}$$

$$\dot{\rho}_{e2} = -\gamma\rho_{e2} - \frac{\mathrm{i}}{2}\left(\Omega_2\rho_{22} - \Omega_2\rho_{ee}\right) - \frac{\mathrm{i}}{2}\Omega_1\rho_{12} \tag{3.23}$$

$$\dot{\rho}_{11} = \gamma\,\rho_{ee} + \frac{\mathrm{i}}{2}\left(\Omega_1\rho_{1e} - \Omega_1^*\rho_{e1}\right) \tag{3.24}$$

$$\dot{\rho}_{22} = \gamma\,\rho_{ee} + \frac{\mathrm{i}}{2}\left(\Omega_2\rho_{2e} - \Omega_2^*\rho_{e2}\right) \tag{3.25}$$

$$\dot{\rho}_{12} = -\mathrm{i}\Delta_1\rho_{12} - \frac{\mathrm{i}}{2}\left(\Omega_1^*\rho_{e2} - \Omega_2\rho_{1e}\right) \tag{3.26}$$

These equations look quite complicated. To simplify them, we shall write them in the basis of the adiabatic states $|a,s\rangle$ that we have introduced before. The transformation between the basis states is given by

$$|1\rangle = \frac{\Omega_1 |s\rangle + \Omega_2^* |a\rangle}{\Omega} \tag{3.27}$$

$$|2\rangle = \frac{\Omega_2 |s\rangle - \Omega_1^* |a\rangle}{\Omega}, \tag{3.28}$$

where $\Omega = \left(|\Omega_1|^2 + |\Omega_2|^2\right)^{1/2}$.

For the Hamiltonian (3.18), we get

$$\begin{aligned} H &= \frac{\hbar\Omega}{2}\left(|e\rangle\langle s| + \text{h.c.}\right) + \hbar\Delta_1 \underbrace{\frac{|\Omega_2|^2}{\Omega^2}}_{d_s} |s\rangle\langle s| + \hbar\Delta_1 \underbrace{\frac{|\Omega_1|^2}{\Omega^2}}_{d_a} |a\rangle\langle a| \\ &\quad + \hbar\Delta_1\Big(\underbrace{\frac{\Omega_1\Omega_2}{\Omega^2}}_{r/2} |s\rangle\langle a| + \text{h.c.}\Big). \end{aligned} \tag{3.29}$$

We see that only the "symmetric" state $|s\rangle$ is coupled to the excited state. For nonzero detuning, symmetric and antisymmetric states acquire AC Stark shifts and are also coupled. This coupling gives rise to the steep refractive index that is interesting for a small group velocity.

The jump Liouvillian (3.20) that is responsible for the loss is transformed as follows (this is a consequence of the unitarity of the transformation into the adiabatic basis):

$$\mathcal{L}_{\text{jump}}[\rho] = \gamma |s\rangle\langle s| \rho_{ee} + \gamma |a\rangle\langle a| \rho_{ee}.$$

Finally, we get the following Bloch equations in the adiabatic basis

$$\begin{aligned} \dot\rho_{ee} &= -2\gamma\rho_{ee} - \frac{\mathrm{i}}{2}\Omega(\rho_{se} - \rho_{es}) \\ \dot\rho_{es} &= (-\gamma + \mathrm{i}\Delta_1 d_s)\rho_{es} - \frac{\mathrm{i}}{2}\Omega(\rho_{ss} - \rho_{ee}) + \frac{\mathrm{i}}{2}\Delta_1 r^*\rho_{ea} \\ \dot\rho_{ea} &= (-\gamma + \mathrm{i}\Delta_1 d_a)\rho_{ea} - \frac{\mathrm{i}}{2}\Omega\rho_{sa} - \frac{\mathrm{i}}{2}\Delta_1 r\rho_{es} \\ \dot\rho_{ss} &= \gamma\rho_{ee} + \frac{\mathrm{i}}{2}\Omega(\rho_{se} - \rho_{es}) - \frac{\mathrm{i}}{2}\Delta_1(r^*\rho_{sa} - r\rho_{as}) \\ \dot\rho_{aa} &= \gamma\rho_{ee} + \frac{\mathrm{i}}{2}\Delta_1(r^*\rho_{sa} - r\rho_{as}) \\ \dot\rho_{sa} &= -\mathrm{i}\Delta_1(d_s - d_a)\rho_{12} - \frac{\mathrm{i}}{2}\Omega\rho_{ea} - \frac{\mathrm{i}}{2}\Delta_1 r(\rho_{aa} - \rho_{ss}) \end{aligned}$$

The Bloch equations, when the steady state is computed numerically for arbitrary detuning, yield a dispersion spectrum with two dispersive resonances located symmetrically around $\Delta_1 = 0$, with a nearly linear variation near resonance. As seen in figure 3.5, the absorption spectrum shows two symmetric peaks, but is zero at resonance. This is because the atomic population is driven into the dark state where it ignores the light field.

Dispersion calculated

We now compute analytically the response of the atom for small detunings. This allows to perform the frequency derivative of the refractive index required in (3.15).

To *zeroth order* in Δ_1, the steady state of the Bloch equations is (you check the details in the exercises)

$$\rho_{aa}^{(0)} = 1,$$

all other matrix elements being zero. This means that the atom is driven into the antisymmetric (or dark) state where it decouples from the laser fields. This happens like in optical pumping.

To *first order*, we get corrections to this solution. The coherences with the antisymmetric state are changed:

$$\rho_{ea}^{(1)} = -\frac{\Delta_1 r}{\Omega} \tag{3.30}$$

$$\rho_{sa}^{(1)} = -\frac{2\mathrm{i}\gamma\Delta_1 r}{\Omega^2} \tag{3.31}$$

The coherence ρ_{es} changes only to second order in Δ_1, as you check in the exercises.

We are now in the position to calculate the dispersion of the three level medium around resonance. According to (3.16), the index is related to the coherence ρ_{e1}. Using the transformation into the adiabatic basis, it is given by

$$\rho_{e1} = \frac{\Omega_1\rho_{es} + \Omega_2^*\rho_{ea}}{\Omega}.$$

To zeroth order in Δ_1, both matrix elements in this equation vanish, and the index is unity at resonance: the medium is transparent ("electromagnetically induced transparency"). To first order in the detuning, we get a

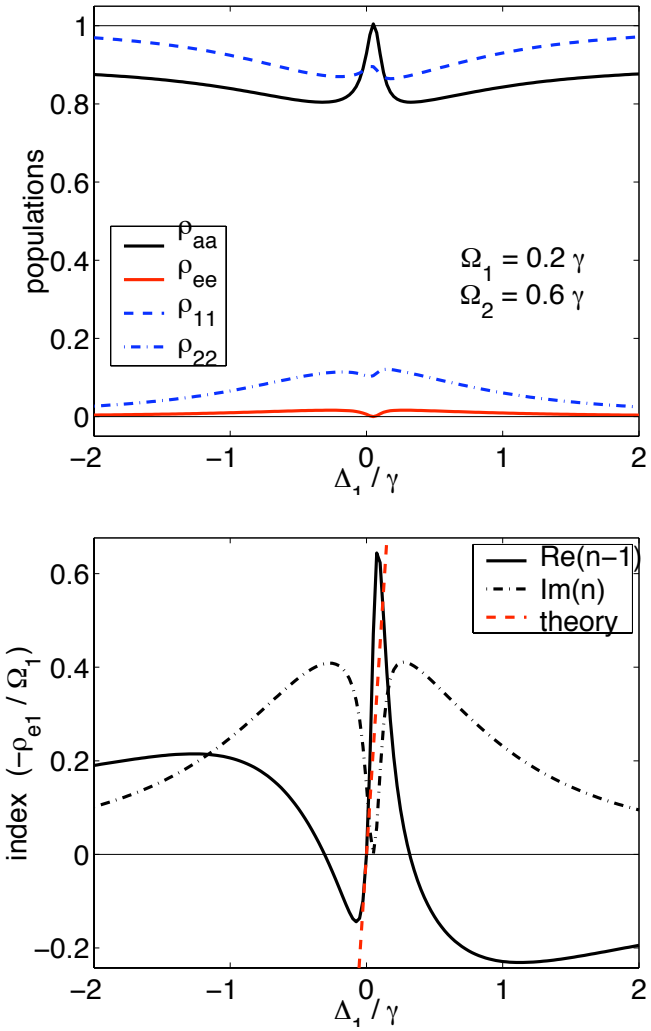

Figure 3.5: Top: steady-state populations vs. probe laser detuning Δ_1 for a three-level atom. Bottom: absorption and dispersion spectrum vs. detuning. The dashed line is the analytical prediction (3.32).

contribution from ρ_{ea} (the coherence ρ_{es} only changes to second order, as you may check in the exercises) and find

$$\rho_{e1}^{(1)} = -\Delta_1 \frac{\Omega_2^* r}{\Omega^2} = -\Delta_1 \frac{2\Omega_1 |\Omega_2|^2}{\Omega^4}. \tag{3.32}$$

We see here the linear response in the both the detuning and the Rabi frequency of the "probe" laser 1. Our final result for the group velocity is

$$\begin{aligned} \frac{c}{v_g} &\approx \omega_1 \frac{\mathrm{d}n}{\mathrm{d}\omega} \\ &= -\frac{D_{e1}^2 N}{\hbar \varepsilon_0} \mathrm{Re}\left(\frac{\Omega_1^*}{|\Omega_1|^2} \frac{\rho_{e1}^{(1)}}{\Delta_1} \right) \\ &= \frac{2D_{e1}^2 N}{\hbar \varepsilon_0} \frac{|\Omega_2|^2}{\Omega^4} \approx \frac{2D_{e1}^2 N}{\hbar \varepsilon_0 |\Omega_2|^2}. \end{aligned} \tag{3.33}$$

In the last line, we have assumed a "weak" probe laser $\Omega_1 \ll \Omega_2$. Note that we then get precisely the prediction (3.17) quoted in the experimental paper.

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