Einführung in die Quantenoptik

Sommersemester 2009
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Übungsaufgaben Blatt 4

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Problem 4.1 – Projekt Wikipedia (5 points)

[Abgabe per Email bis zum 16. Juni 2009]

Schreiben Sie einen Absatz Wikipedia-Fließtext, in dem die "elektrische Dipol-Kopplung" für Atome beschrieben und qualitativ begründet wird. Stellen Sie einen Minimalsatz an relevanten Formeln zusammen und schreiben Sie diese im Wikipedia-Format auf.

Problem 4.2 – Thermal radiation field (8 points)

The concept of a thermal state combines stationary quantum states with classical statistics in the canonical ensemble. Each stationary state $|\{n\}\rangle$ with energy $E_{\{n\}}$ appears with a weight given by the Boltzmann factor $\exp(-E_{\{n\}}/k_{\rm B}T)$. The notation $\{n\} = \{n_1, \ldots, n_k, \ldots\}$ stands for the set of occupation (photon) numbers in all field modes.

(i) Show first that for a stationary state, one has

$$\langle\{n\}|a_k^\dagger a_l|\{n\}\rangle = \delta_{kl} n_k, \qquad \langle\{n\}|a_k a_l^\dagger|\{n\}\rangle = \delta_{kl}(n_k + 1) \tag{4.1}$$

where n_k is the photon number in mode k. Show also that all other products of one or two mode operators have zero expectation value.

(ii) Now consider the canonical average with respect to the Boltzmann weight and show these expectation values are changed into

$$\langle a_k^\dagger a_l\rangle_T = \delta_{kl}\bar{n}(\omega_k), \qquad \langle a_k a_l^\dagger\rangle = \delta_{kl}(\bar{n}(\omega_k) + 1) \tag{4.2}$$

where $\bar{n}(\omega_k) = ({\rm e}^{\beta\omega_k} - 1)^{-1}$ is the mean photon number for a mode at frequency ω_k. (Notation: $\beta\omega_k = \hbar\omega_k/(k_{\rm B}T)$.)

(iii) In the lecture, we encountered the fluctuation spectrum of the thermal radiation field that shows qualitatively the behaviour (C is a constant, note the exponent that differs from the lecture)

$$S(\omega) = C\omega^3\bar{n}(\omega) \tag{4.3}$$

Discuss its behaviour at negative frequencies in the domains $1 \gg -\beta\omega > 0$, $-\beta\omega \sim 1$, and $-\beta\omega \gg 1$. Does this curve show an inflexion point (*Wendepunkt*) somewhere, as conjectured in the lecture?

Problem 4.3 – Fluctuation spectra and operator ordering (XX points)
Consider the field operator

$$E(t) = \sum_k g_k a_k(t) + \text{h.c.} \tag{4.4}$$

where the g_k are complex constants. (i) A convenient way to count the number of modes is given by the commutator

$$C(\tau, t) := [E(t+\tau),\, E(t)] \tag{4.5}$$

Show that this is a complex-valued function of τ (not an operator) whose Fourier transform $J(\omega)$ is (what Fourier sign convention?)

$$J(\omega) = 2\pi \sum_k |g_k|^2 \left(\delta(\omega - \omega_k) - \delta(\omega + \omega_k)\right) \tag{4.6}$$

(ii) In the lecture, we have discussed the spectrum $S(\omega)$ corresponding to (non-ordered) correlation functions like $\langle E(t+\tau)E(t)\rangle$. Prove in thermal equilibrium the 'fluctuation–dissipation relation'

$$S(\omega) = \bar{n}(\omega) J(\omega), \tag{4.7}$$

valid for both positive and negative frequencies. (Use the continuation of $\bar{n}(\omega)$ to negative arguments discussed in the lecture.)

(iii) Prove a similar relation for symmetrically ordered correlations

$$\tfrac{1}{2}\langle E(t+\tau)E(t) + E(t)E(t+\tau)\rangle = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \left(\tfrac{1}{2} + \bar{n}(\omega)\right) J(\omega)\, \mathrm{e}^{\mathrm{i}\omega\tau} \tag{4.8}$$

Check that the integrand $\left(\frac{1}{2} + \bar{n}(\omega)\right) J(\omega)$ is symmetric (even) in ω so that indeed the integral becomes real.

Problem 4.4 – Coherence and spectrum (6 points)
The (first order) coherence function of a field is defined by

$$g^{(1)}(t, t') = \langle E^{(-)}(t) E^{(+)}(t')\rangle. \tag{4.9}$$

where $E^{(\pm)}(t)$ are the positive (negative) frequency components of the field operator (4.4) (i.e., $E^{(+)}(t)$ contains only annihilation operators, $E^{(-)}(t)$ only creation operators).

The field (more precisely: the state of the field) is called *stationary* if $g^{(1)}$ only depends on the time difference $t - t'$. It is called *first-order coherent* if $g^{(1)}(t, t')$ factorizes: this means that a function $\mathcal{E}(t)$ exists such that $g^{(1)}(t, t') = \mathcal{E}^*(t)\mathcal{E}(t')$. The *spectrum* of a stationary field is defined by the Fourier transform of $g^{(1)}(t - t')$ with respect to $t - t'$.

(a) Show that a single-mode field is first-order coherent.

(b) Show that a stationary, first-order coherent field is monochromatic.