Quanten-Informatik und theoretische Quantenoptik II

Sommersemester 2010

Carsten Henkel

Problem set 1 Hand out: 22 April 2010 hand in: 29 April 2010

Problem 1.1 – Lindblad operators and Heisenberg picture (5 points)

In the lecture, you have seen the 'Lindblad (super)operator' that describes the non-Schrödinger dynamics for a two-level system (density matrix ρ) subject to spontaneous emission (rate γ):

$$\mathcal{L}(\rho) = \gamma\sigma\rho\sigma^\dagger - \frac{\gamma}{2}\left\{\rho\sigma^\dagger\sigma + \sigma^\dagger\sigma\rho\right\} \tag{1.1}$$

Now, in the Heisenberg picture, you are interested in the expectation value of a system operator, $\langle A\rangle_t = \mathrm{tr}[A(t)\rho] = \mathrm{tr}[A\rho(t)]$. Show from (1.1) that this expectation value satisfies the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle_t = \frac{\mathrm{i}}{\hbar}\langle[H,A]\rangle_t + \frac{\gamma}{2}\langle[\sigma^\dagger, A]\sigma + \sigma^\dagger[A,\sigma]\rangle_t \tag{1.2}$$

• Consider an observable $C = \mathrm{i}[A,B]$ and its equation of motion in the Heisenberg picture. Under Hamiltonian evolution, $C(t)$ remains the commutator of $A(t)$ and $B(t)$. Does this remain true (in expectation value) for an open (dissipative) system? Is this a problem? Start playing by removing the brackets $\langle\cdots\rangle$ from Eq.(1.2). [5 bonus points]

Problem 1.2 – Bloch equations (10 points)

(a) Starting from the Lindblad (super)operator, derive the equations of motion for the density matrix elements $\rho_{ab} = \langle a|\rho|b\rangle$ $(a, b = \mathrm{e}, \mathrm{g})$ of a two-level atom driven by a classical laser field,

$$H_{\mathrm{AL}} = \frac{\hbar(\omega_A - \omega_L)}{2}\sigma_3 + \frac{\hbar}{2}(\Omega^*\sigma + \Omega\sigma^\dagger) \tag{1.3}$$

where Ω is the Rabi frequency and $\Delta = \omega_L - \omega_A$ the detuning.

• Eq.(1.3) applies only in the rotating frame where the density operator is written in the form $\exp(-i\omega_L t\sigma_3)\rho(t)\exp(i\omega_L t\sigma_3)$. Check that the Lindblad superoperator takes in this frame the same form as in Eq.(1.1). [5 bonus points]

(b) Show that the stationary solution to the density matrix has a form similar to (no guarantee for signs and numerical factors)

$$\rho_{\mathrm{ge}} = \frac{\Omega}{2}\frac{\Delta - \mathrm{i}\gamma/2}{\Delta^2 + \gamma^2/4 + |\Omega|^2/2} \tag{1.4}$$

$$\rho_{\mathrm{ee}} = \frac{|\Omega|^2/4}{\Delta^2 + \gamma^2/4 + |\Omega|^2/2} \tag{1.5}$$

In the complex plane of the laser amplitude, $\mathcal{E}(t) \sim -\Omega\,\mathrm{e}^{-\mathrm{i}\omega_L t}$ (ignoring polarization), and of the atomic dipole $d(t) \sim \sigma\,\mathrm{e}^{-\mathrm{i}\omega_L t}$ (ignoring the orientation of $\mathbf{d}_{\mathrm{ge}}$), what is the 'phase lag' between these two amplitudes?

Problem 1.3 – Non-positive master equation (5 points)

In certain systems, in particular in the condensed phase, diagonal and off-diagonal elements of the density matrix decay with different rates:

$$\frac{\mathrm{d}\rho_{\mathrm{ee}}}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}\langle \mathrm{g}|[H,\rho]|\mathrm{e}\rangle - \gamma\rho_{\mathrm{ee}} \tag{1.6}$$

$$\frac{\mathrm{d}\rho_{\mathrm{ge}}}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}\langle \mathrm{g}|[H,\rho]|\mathrm{e}\rangle - \Gamma\rho_{\mathrm{ge}} \tag{1.7}$$

where the equations for ρ_{gg} and ρ_{eg} follow from trace conservation and hermiticity, respectively. Let us call '*decoherence rate*' the rate Γ for the coherences (off-diagonal elements) and '*decay rate*' the rate γ. Show that the decoherence rate satisfies the inequality

$$\Gamma \geq \frac{\gamma}{2} \tag{1.8}$$

which is often written in the form $T_2 = 1/\Gamma \leq 2T_1 = 2/\gamma$.

Hint. Calculate the time derivative $\dot{P}$ of the purity

$$P = \mathrm{Pu}(\rho) = \mathrm{tr}(\rho^2 - \rho), \tag{1.9}$$

look for the pure state that maximizes it, and calculate $\dot{P}$ for this state. This maximum value should be zero or negative. • Why? [3 bonus points]