Quanten-Informatik und theoretische Quantenoptik II

Sommersemester 2010

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Problem set 3 Hand out: 21 May 2010 hand in: 02 Jun 2010

Problem 3.1 – P- and Q-functions (5 points)

In the lecture, we mentioned the following properties of the (Glauber-) P- and (Husimi-) Q-function and the thermal equilibrium state:

$$\begin{aligned} Q(\alpha) &= \int \frac{\mathrm{d}\beta}{\pi} \mathrm{e}^{-|\alpha-\beta|^2} P(\beta) && (3.1) \\ P_T(\alpha) &= \frac{\mathrm{e}^{-|\alpha|^2/\bar{n}}}{\pi \bar{n}}, \quad \text{(thermal photon number } \bar{n}) && (3.2) \\ \langle D(\alpha) \rangle_T &= \mathrm{e}^{-|\alpha|^2(\bar{n}+\frac{1}{2})} && (3.3) \end{aligned}$$

Prove these equations. Work out the expectation value

$$\langle n|D(\alpha)|n\rangle = L_n(|\alpha|^2)\,\mathrm{e}^{-\frac{1}{2}|\alpha|^2} \tag{3.4}$$

where $L_n(\cdot)$ is the Laguerre polynomial (look up its definition and properties in Wikipedia).

Problem 3.2 – Dephasing rate in the Markov limit (5 points)

In the lecture, we have discussed the dephasing factor $\mathrm{e}^{-\Gamma(t)}$ for a two-level system. Calculate the asymptotic (at large times) dephasing rate

$$\gamma = \lim_{t\to\infty} \frac{\mathrm{d}\Gamma}{\mathrm{d}t} \propto \alpha T \tag{3.5}$$

from the integral given in the lecture and an Ohmic spectral density

$$\Gamma(t) = \int_0^\infty \mathrm{d}\omega\, \frac{\sin^2(\frac{1}{2}\omega t)}{\omega^2} \frac{8\alpha\omega\omega_c^2}{\omega_c^2+\omega^2} \coth\frac{\omega}{2T} \tag{3.6}$$

Hint. One possibility is to write the integral for $\mathrm{d}\Gamma/\mathrm{d}t$ as the imaginary part of a complex function, to rotate the integration path by some angle into the complex frequency plane and to expand the integrand around $\omega = 0$.

Problem 3.3 – Non-Markovian master equation (5 points)

The Lindblad equation derived in the lecture has the "Markov property", meaning that the time derivative of the system density operator, $\mathrm{d}\rho/\mathrm{d}t$, depends only on the present time t. In Nature, "memory effects" will occur, however, which require a "non-Markovian" description.

In the literature, one attempt towards a non-Markovian master equation is to use time-dependent expressions for the decay rate and the frequency shift:

$$\gamma(t) + \mathrm{i}\delta\omega_{\mathrm{A}}(t) = \int_0^t \mathrm{d}\tau\, C(\tau)\, \mathrm{e}^{\mathrm{i}\omega_{\mathrm{A}}\tau} \tag{3.7}$$

where $C(\tau)$ is the correlation function of some environment variables

$$C(\tau) = \langle B(\tau)B^{\dagger}(0)\rangle \tag{3.8}$$

and the average is taken with respect to some equilibrium state for the bath. Consider the following "non-Markovian master equation" for the dipole operator

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\left[\gamma(t) + \mathrm{i}\omega_{\mathrm{A}} + \mathrm{i}\delta\omega_{\mathrm{A}}(t)\right]\sigma \tag{3.9}$$

and *show* that it has an exact solution where $|\langle\sigma(t)\rangle| \leq |\langle\sigma(0)\rangle|$ decays in time. *Show* that for short enough times (can you find a quantitative estimate?), the decay is quadratic (not linear) in t.

Problem 3.4 – Conservation of trace and of probability amplitudes (5 points)

Consider first the extension of the dynamical map Λ to "skew operators", mentioned in the lecture:

$$\Lambda(|\psi\rangle\langle\chi|) := \frac{1}{2}\left[\Lambda(\rho_{+1}) - \Lambda(\rho_{-1}) + \mathrm{i}\,\Lambda(\rho_{+\mathrm{i}}) - \mathrm{i}\,\Lambda(\rho_{-\mathrm{i}})\right] \tag{3.10}$$

$$\rho_u = \frac{1}{2}\left(|\psi\rangle + u|\chi\rangle\right)\left(\langle\psi| + u^*\langle\chi|\right), \qquad |u| = 1 \tag{3.11}$$

Compute the trace of ρ_u and *prove* that

$$\mathrm{tr}\,\Lambda(|\psi\rangle\langle\chi|) = \langle\chi|\psi\rangle \tag{3.12}$$

so that the extended dynamical map preserves probability amplitudes.