Quanten-Informatik und theoretische Quantenoptik II

Sommersemester 2010

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Problem set 5 Hand out: 22 Jun 2010 hand in: 28 Jun 2010

Problem 5.1 – Fokker-Planck theory for a damped mode (10 points)

Calculate the evolution of a damped harmonic oscillator prepared initially in a coherent state. This requires to solve the Fokker-Planck equation with the initial value

$$P(\alpha, t) = \delta^2(\alpha - \alpha_0). \tag{5.1}$$

The Fokker-Planck equation reads

$$\partial_t P(\alpha, t) = -\left[(-\mathrm{i}\omega_c - \kappa/2)\partial_\alpha \alpha + (\mathrm{i}\omega_c - \kappa/2)\partial_{\alpha^*}\alpha^*\right] P(\alpha, t) + \kappa \bar{n}(\omega_c)\partial_\alpha \partial_{\alpha^*} P(\alpha, t) \tag{5.2}$$

where ω_c is the mode frequency, κ the damping rate and $\bar{n}(\omega_c)$ the average occupation number of the environment.

It may be easier to consider a gaussian distribution in phase space,

$$P(\alpha, t) = \frac{1}{\pi\sigma^2(t)} \exp\left(-\frac{|\alpha - \beta(t)|^2}{\sigma^2(t)}\right). \tag{5.3}$$

and to derive equations of motion for $\beta(t)$ and $\sigma(t)$. Give a physical interpretation of $\beta(t)$ and $\sigma(t)$. What is the trajectory of the system in the complex α-plane? Consider the limits of strong damping ($\kappa \gg \omega_c$) and of strong diffusion ($\bar{n}(\omega_c)\kappa \gg \omega_c \gg \kappa$).

Problem 5.2 – Quantum Langevin equations (10 points)

A representation of dissipative quantum dynamics that is useful to calculate correlation functions and noise spectra are the so-called quantum Langevin equations. The simplest case is that of a single damped mode with annihilation operator $a(t)$ (Heisenberg picture) that satisfies the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}a = -(\kappa + \mathrm{i}\omega_c)a + \sqrt{\kappa}\,\xi(t) \tag{5.4}$$

where ω_c is the mode frequency and κ the damping rate. The "quantum Langevin force" $\xi(t)$ is an operator-valued random variable with the properties

$$\langle \xi(t) \rangle = 0, \qquad [\xi(t),\, \xi^\dagger(t')] = \delta(t - t') \tag{5.5}$$

(i) *Prove* that the Langevin noise ensures that the equal-time commutation relations have, at all times, the canonical form

$$[a(t), a^\dagger(t)] = 1 \tag{5.6}$$

(ii) Assuming that the Langevin noise has normally ordered correlations of the form

$$\langle \xi^\dagger(t)\xi(t')\rangle = S_\xi\, \delta(t - t'), \tag{5.7}$$

calculate the correlation spectrum $S(\omega)$ in the stationary limit,

$$S(\omega) = \lim_{t\to\infty} \int_{-t}^{t} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega\tau} \langle a^\dagger(t+\tau)a(t)\rangle \sim \frac{\kappa S_\xi}{\kappa^2 + (\omega - \omega_c)^2}. \tag{5.8}$$

Hint. Fourier transform. **5 Bonus points** for the commutator $\langle [a(t), a^\dagger(t')]\rangle$ and its Fourier transform. Compare to Eq.(5.8).