Einführung in die Quantenoptik II Sommersemester 2011

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Problem set 3 Hand out: 11 May 2011 hand in: 25 May 2011

Problem 3.1 – Bath correlation and spectrum (6 points)

In the derivation of the master equation, we have come across autocorrelation functions of bath variables. (i) Take a bosonic bath in a stationary state ρ_B with respect to the bath Hamiltonian H_B and show that $\langle B(t)B(t')\rangle = \mathrm{tr}(B(t)B(t')\rho_B)$ depends only on the time difference $t-t'$ (i.e., the correlation is stationary as well). (ii) For a bath operator $B = \sum_k g_k b_k + \mathrm{h.c.}$ (b_k: annihilation operator) and the bath spectral density

$$J(\omega) = 2\pi \sum_k |g_k|^2 \delta(\omega - \omega_k) \tag{3.1}$$

show that the Fourier transform of $\langle B(t)B(t')\rangle$ is proportional to $J(\omega)$. What other information is needed? Work out the dependence on $t - t'$ for a thermal bath with $J(\omega) = \alpha\,\omega^3$ (real, positive α).

Problem 3.2 – Blowing-up and tracing out (6 points)

We have encountered in the derivation of the master equation the following construction:

$$\rho(t) \mapsto \rho(t + \Delta t) = \mathrm{tr}_B \left[U_{SB}(\Delta t)(\rho(t) \otimes \rho_B)U_{SB}^\dagger(\Delta t)\right] \tag{3.2}$$

which is sometimes known as the Nakajima-Zwanziger projection. Here $U_{SB}(\Delta t)$ is the time evolution operator under the full system+bath Hamiltonian. Show that this prescription defines a completely positive map.

Construct the Kraus operators in a basis where ρ_B is diagonal. → Book by Nielsen & Chuang on quantum information.

Problem 3.3 – Lindblad dynamics for a damped cavity mode (8 points)

(i) Consider a master equation in Lindblad form with a set of Lindblad operators L_k, $L_k^\dagger$ as derived in the lecture. Show that the equation of motion for the average of a system operator A is

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle = \mathrm{i}\langle[H,\, A]\rangle + \frac{1}{2}\sum_k \langle[L_k^\dagger,\, A]L_k + L_k^\dagger[A, L_k]\rangle \tag{3.3}$$

where the expectation values $\langle\cdots\rangle$ are taken in the usual way with respect to the density operator.

The Lindblad equation given in Wikipedia is slightly more general, we took the diagonal form that appears later in this entry.

(ii) Specialize to the case of a single bosonic mode (frequency ω_c) and Lindblad operators for the coupling to a thermal environment,

$$L_{\rm loss} = \sqrt{\gamma\,(\bar{n}+1)}\,a, \qquad L_{\rm abs} = \sqrt{\gamma\,\bar{n}}\,a^\dagger \tag{3.4}$$

where $\bar{n}$ is the average occupation number. Show that the "dissipative Heisenberg equations" for a and for the number operator n take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle a\rangle = -\mathrm{i}\omega_c\langle a\rangle - \frac{\kappa}{2}\langle a\rangle \tag{3.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle n\rangle = \kappa\,\bar{n} - \kappa\langle n\rangle \tag{3.6}$$

Conclude that the stationary solution corresponds to a thermal state at the temperature of the environment.

(iii) The algebraic structure in the equation of motion (3.3) is more general than the commutators familiar from quantum mechanics. If one tries to use it on a vector space of operators (i.e., without taking expectation values), one may ask the question what is the minimum set of operators that is closed under this dynamics. Explore this set for a single bosonic mode (operators a, $a^\dagger$) and for a two-level system (Pauli spin operators σ_1, σ_2, σ_3).

In ordinary quantum mechanics, the set $\{a,\, a^\dagger,\, \mathbb{1}\}$ is sufficient. For a mode with dissipation, the set is larger.