Einführung in die Quantenoptik II Sommersemester 2011

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Problem 5.1 – Photon statistics of a laser (10 points)

In the lecture, we discussed the quantum theory of the laser in the formulation of Scully & Lamb. The photon statistics $p(n)$ is given by the recurrence relation

$$p(n+1) = \frac{G/\kappa}{1 + Bn} p(n) \tag{5.1}$$

(1) Give the physical meaning of the parameters G, κ, B. (2) Evaluate the photon statistics numerically and make a plot for typical parameter values $1/B = 20$, $G/\kappa = 0.1$ and $1/B = 30$, $G/\kappa = 1.3$. (3) Make a plot of the average photon number $\langle n \rangle$ and its normalized variance $Q = \Delta n^2 / \langle n \rangle$ vs the parameter G/κ for $1/B = 25$. (4) Show that the recurrence relation can be solved with the help of the gamma function $\Gamma(n)$, using the formula

$$\Gamma(z+1) = z\Gamma(z) \tag{5.2}$$

Use the Stirling expansion for large arguments to find a gaussian approximation to the photon statistics when the average photon number $\langle n \rangle$ is large. What do you get for the variance? Is that consistent with your numerical solution?

Problem 5.2 – Correlation functions and spectra (4 points)

For a non-hermitean operator $\hat{a}(t)$ define formally the Fourier transform as

$$\tilde{a}(\omega) = \int \mathrm{d}t \, \mathrm{e}^{\mathrm{i}\omega t} \hat{a}(t) \tag{5.3}$$

and (1) show that the average spectrum is given by

$$\langle \tilde{a}^\dagger(\omega) \tilde{a}(\omega) \rangle = 2\pi \delta(\omega - \omega') S_a(\omega) \tag{5.4}$$

where $S_a(\omega)$ is the Fourier transform of the autocorrelation function $\langle \hat{a}^\dagger(t) \hat{a}(t') \rangle$ with respect to the time difference $t - t'$. (2) Assume that the operator $\hat{a}(t)$ can be approximated by $\alpha \, \mathrm{e}^{-\mathrm{i}\omega_L t} + \delta\hat{a}(t)$ with $\langle \delta\hat{a}(t) \rangle = 0$. What do you get for the for the spectrum?

Problem 5.3 – Jaynes-Cummings-Paul dynamics and closed sub-spaces (6 points)

Consider a two-level system coupled to a single radiation mode via the resonant interaction

$$H = \hbar\omega_A \sigma^\dagger \sigma + \hbar\omega_L a^\dagger a + \hbar(g^* a^\dagger \sigma + g\sigma^\dagger a) \tag{5.5}$$

Recall that the total excitation $N = \sigma^\dagger\sigma + a^\dagger a$ commutes with the Hamiltonian and one can restrict the dynamics to the subspace spanned by the two eigenvectors of N with eigenvalue n. (1) Identify these eigenvectors. (2) Calculate the matrix elements of H in the basis of these eigenvectors. (3) Calculate the time-evolution operator $U(\tau) = \exp(-\mathrm{i}H\tau)$ in this basis by writing the 2×2 matrix representation of H as a sum of Pauli matrices (take real g for simplicity) and removing the term proportional to the unit operator. Check that one gets a result consistent with the formula given in the lecture (I copy this formula from the book by M. Orszag):

$$U(\tau) = \begin{pmatrix} \cos\tau\varphi + \mathrm{i}\frac{\Delta\tau}{2}\,\mathrm{sinc}\,\tau\varphi & -\mathrm{i}g\tau\,\mathrm{sinc}(\tau\varphi)a \\ -\mathrm{i}g\tau a^\dagger\,\mathrm{sinc}\,\tau\varphi & \cos\tau\varphi_- - \mathrm{i}\frac{\Delta\tau}{2}\,\mathrm{sinc}\,\tau\varphi_- \end{pmatrix} \tag{5.6}$$

where $\mathrm{sinc}(x) = \sin(x)/x$, $\varphi = \sqrt{g^2 a a^\dagger + \Delta^2/4}$, $\varphi_- = \sqrt{g^2 a^\dagger a + \Delta^2/4}$, and $\Delta = \omega_L - \omega_A$.