Einführung in die Quantenoptik II

Sommersemester 2011

Carsten Henkel

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Problem 6.1 – Mollow spectrum (12 points)

We have seen that the autocorrelation function of the dipole operator of a two-level atom can be calculated with the help of the Bloch equations by introducing the skew operator $P(\tau|t,\sigma)$

$$\lim_{t\to\infty} \langle \sigma^\dagger(t+\tau)\sigma(t)\rangle = \lim_{t\to\infty} \mathrm{tr}[\sigma^\dagger P(\tau|t,\sigma)] \tag{6.1}$$

$P(\tau|\sigma)$ satisfies the master equation with the initial condition $P(\tau|t,\sigma) = \sigma\rho(t) \to \sigma\rho_{\mathrm{st}}$ where ρ_{st} is the stationary state. Expand $P(\tau|t,\sigma)$ in Pauli matrices,

$$P(\tau|t,\sigma) = \frac{1}{2}\left\{ p_0 \mathbb{1} + \sum_{j=1}^{3} p_j(\tau)\sigma_j \right\} \tag{6.2}$$

and (1) calculate the coefficient p_0. (2) Why does it not depend on τ? The Bloch equations are

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -(\Gamma - \mathrm{i}\Delta)p + \mathrm{i}(\Omega/2)p_3 \tag{6.3}$$

$$\frac{\mathrm{d}p_3}{\mathrm{d}t} = -\gamma(p_3 + 1) + \mathrm{i}(\Omega^* p - \Omega p^*) \tag{6.4}$$

where $p = (p_1 - \mathrm{i}p_2)/2$, Ω is the Rabi frequency, $\Delta = \omega_L - \omega_A$ the laser detuning, Γ the decay rate for optical coherences, and γ the spontaneous decay rate of the excited state. These equations can be written as a system of three real differential equations with constant coefficients. (3) Show that the "secular equation" (i.e., the polynomial whose roots λ are the complex decay constants) is given by (allowed simplification: consider real Ω)

$$D(\lambda) = \lambda^3 - (2\Gamma + \gamma)\lambda^2 + (\Delta^2 + 2\Gamma\gamma + \Gamma^2)\lambda - (\Gamma^2\gamma + \Gamma|\Omega|^2 + \gamma\Delta^2) \tag{6.5}$$

(4) Make a plot of this polynomial for typical parameters (consider "weak" and "strong" laser fields). When does one get a pair of complex conjugate zeros? (5) Use the rule of Viétà to show the three zeros λ_j of $D(\lambda)$ satisfy the following sum rule $\lambda_1 + \lambda_2 + \lambda_3 = \gamma + 2\Gamma$. (5) [5 Bonus points:] discuss the right eigenvector of the Bloch equations coming with one of the complex eigenvalues in the limit of a strong laser field. Compare this state to the dressed states, i.e., the eigenvectors of the atom+laser Hamiltonian. Allowed simplification: $\Delta = 0$, $\Gamma = \gamma/2$.

Problem 6.2 – Quantising a light beam (8 points)

In the lecture, we have formulated field quantization in a "box" (volume $V = L^3$ with periodic boundary conditions). A light beam in the laboratory looks different. Consider, as an alternative scheme, plane-wave mode functions with frequency $\omega = 2\pi ck/L$ (integer k). For simplicity, we start in one dimension. Construct the operators

$$a(\omega) = \sqrt{\frac{L}{c}} a_k \tag{6.6}$$

and (1) show that their commutation relation becomes for $L \to \infty$ (**Hint**: compare the weight of the δ-function by switching between discrete and continuous frequencies)

$$\left[a(\omega), a^\dagger(\omega')\right] = 2\pi\,\delta(\omega - \omega'). \tag{6.7}$$

Consider the "narrow-band" field operator (ω_0 is a reference frequency)

$$E(t) = \sqrt{\frac{\hbar\omega_0}{2\varepsilon_0}} \int \frac{\mathrm{d}\omega}{2\pi} a(\omega)\, \mathrm{e}^{-\mathrm{i}\omega t} \tag{6.8}$$

and (2) show that the commutation relations in the time domain become

$$\left[E(t), E^\dagger(t')\right] = \frac{\hbar\omega_0}{2\varepsilon_0} \delta(t - t') \tag{6.9}$$

A field operator with this property is called "quantum noise".

(3) Consider the autocorrelation function $\langle E^\dagger(t+\tau)E(t)\rangle$ and find a necessary and sufficient condition on the expectation value of the $a(\omega)$ operators ensuring that the autocorrelation depends only on the time difference τ.

(4) Pick a quadrature $X_\theta(t)$ of the field operator $E(t)$ and calculate its autocorrelation function for a field in the vacuum state. The corresponding spectrum is called "shot-noise". (**Reminder**: a quadrature $X_\theta(t)$ is a hermitean operator constructed from a linear combination of $E(t)$ and $E^\dagger(t)$.)