Einführung in die Quantenoptik II

Sommersemester 2012

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Problem set 2 Hand out: Fri 20 Apr 2012 hand in: Wed 04 May 2012

Problem 2.1 – Kraus operators (8 points)

In the lecture, we have seen the Kraus representation for a completely positive map

$$T[\rho] = \sum_k \Omega_k \rho \Omega_k^\dagger \tag{2.1}$$

and the construction of Kraus operators Ω_k. (i) *Show* that the preservation of the trace is equivalent to

$$\sum_k \Omega_k^\dagger \Omega_k = \mathbb{1} \tag{2.2}$$

where $\mathbb{1}$ is the unit operator on the system Hilbert space. This can be understood as a completeness relation or normalization.

(ii) In Exercise 1.3(2), you have seen the Hilbert-Schmidt scalar product between operators $(A, B) = \mathrm{tr}(A^\dagger B)$. A scalar product is called non-degenerate when the following property holds: $(A, B) = 0$ for all B if and only if $A = 0$. Show that this property is needed to prove (i) because from $\mathrm{tr}(K\rho) = \mathrm{tr}(\rho)$, valid for all density operators ρ, one can conclude that $K = \mathbb{1}$. Do you find a loophole somewhere?

(iii) *Show* that the Kraus operators constructed in the proof of the Kraus theorem (see lecture notes) satisfy Eq.(2.2). Think about the normalization of the Choi matrix based on the maximally entangled state $|\phi\rangle \sim \sum_n |n \otimes n\rangle$, that we did not check in the proof.

Problem 2.2 – Random unitaries (5 points)

Consider a set of unitary operators U_k ($k = 1, \ldots N$) and positive numbers p_k. (i) Under which conditions does

$$\rho \mapsto T[\rho] = \sum_k p_k U_k \rho U_k^\dagger \tag{2.3}$$

implement a dynamical (completely positive) map? Give a physical interpretation of this map.

(ii) Start with a pure state $\rho = |\psi\rangle\langle\psi|$, apply the map (2.3) and calculate the von Neumann entropy of $T[\rho]$:

$$S(T[\rho]) = -\mathrm{tr}\{T[\rho] \log T[\rho]\} \tag{2.4}$$

You many specialize to a two-level system and a particular choice of the U_k to get forward in the calculation. [5 bonus points]

(ii) Check whether the previous construction can be generalized to

$$\rho \mapsto \sum_{kl} b_{kl} U_k \rho U_l^\dagger \tag{2.5}$$

where b_{kl} is a positive matrix (hermitean?). [5 bonus points]

Problem 2.3 – On the Stinespring dilation theorem (7 points)

(i) Consider a quantum system in a mixed state (density operator ρ). The Stinespring dilation theorem says that this can be related to a pure state in a larger ("dilated") Hilbert space, Analyze how this construction works: take the eigenvectors $|\psi_n\rangle$ and eigenvalues p_n of ρ, add a "quantum notepad" with states $|n\rangle$ and construct the state vector

$$|\Psi\rangle = \sum_n \sqrt{p_n} |\psi_n \otimes n\rangle \tag{2.6}$$

What is the reduced density operator that corresponds to $|\Psi\rangle$? What kind of measurements would be needed to tell the difference between the states ρ and $|\Psi\rangle\langle\Psi|$?

(ii) In quantum information, one sometimes identifies density operators with completely positive maps. More precisely, for each ρ there is a map T and a density operator ρ_0 such that $T[\rho_0] = \rho$. To see this, consider the expansion of a mixed state into its eigenvectors

$$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n| \tag{2.7}$$

Try to construct a map T by taking the Kraus operators Ω_k proportional to the projectors $|\psi_n\rangle\langle\psi_n|$ and pick an initial density operator ρ_0, such that $T[\rho_0] = \rho$.