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**Introduction to Quantum Optics II**

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The preliminary programme for this lecture:

1. Axiomatic theory of quantum evolutions  $\hat{\rho}(0) \mapsto \hat{\rho}(t)$ 
  - Kraus-Stinespring theorem, Lindblad theorem
  - examples: spontaneous decay, dephasing, lossy cavity, micro-maser
    - talks: Monte Carlo technique, radiation forces and laser cooling
2. Fluctuations and spectra
  - correlations, fluctuations, spectral characterization
  - quantum regression formula (“theorem”). Example: resonance fluorescence
  - quantum Langevin equations, input-output approach. Example: squeezing in a cavity
  - fluctuation–dissipation theorem
    - talks: regression formula and squeezing, Hanbury Brown–Twiss correlations and anti-bunching
3. Quantum theory of the laser
  - Scully-Lamb theory: rate equations, photon statistics
  - linewidth of the laser (Schawlow-Townes): Fokker-Planck equation
  - micromaser
    - talks: covariant QED, Rydberg atoms

# Chapter 0

## Reminder: atom-field interactions

### 0.1 Hamiltonian and relevant approximations

Hamiltonian  $H_A + H_F + H_{AF}$

$$H_A = E_e|e\rangle\langle e| + E_g|g\rangle\langle g| = \frac{\hbar\omega_A}{2}\sigma_3, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1)$$

two relevant atomic states  $|g\rangle$  and  $|e\rangle$ , transition matrix element of the electric dipole operator  $\mathbf{d}_{ge} = \langle g|\hat{\mathbf{d}}|e\rangle$ , two-level annihilation operator  $\sigma = |g\rangle\langle e|$

Coupling to the electromagnetic field via the electric dipole moment, in resonance (“rotating wave”) approximation

$$H_{AF} = -\hat{\mathbf{d}}\cdot\mathbf{E}(\mathbf{x}_A, t) \approx -\left(\mathbf{d}_{eg}\sigma^\dagger \cdot \hat{\mathcal{E}}(\mathbf{x}_A, t) + \text{h.c.}\right), \quad \sigma = |g\rangle\langle e| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (2)$$

Split of the electric field into positive and negative frequency components

$$\hat{\mathbf{E}}(\mathbf{x}_A, t) = \underbrace{\hat{\mathcal{E}}(\mathbf{x}_A, t)}_{a_k e^{-i\omega_k t}} + \underbrace{\hat{\mathcal{E}}^\dagger(\mathbf{x}_A, t)}_{a_k^\dagger e^{i\omega_k t}} \quad (3)$$

Expansion in modes with quantum number  $k$ , (classical) mode functions  $\mathbf{f}_k(\mathbf{x})$ , annihilation operator  $a_k(t)$ , one-photon field amplitude  $\mathcal{E}_k = (\hbar\omega_k/2\varepsilon_0)^{1/2}$ .

Classical approximation (laser field):  $\hat{\mathcal{E}} \mapsto \mathcal{E}_L(t)$  complex amplitude. In the quantum theory: field (mode) in coherent state  $a_k|\alpha_k\rangle = \alpha_k|\alpha_k\rangle$ . Time dependence of the field in quantum theory: via field Hamiltonian

$$H_F = \sum_k \hbar\omega_k \left( a_k^\dagger a_k + \frac{1}{2} \right) \quad (4)$$

one-photon coupling frequency

$$\hbar g_k = -\mathcal{E}_k \mathbf{d}_{ge} \cdot \mathbf{f}_k(\mathbf{x}_A) \quad (5)$$

## Approximations

Electric dipole approximation: atom size small compared to “relevant wavelengths”.

Resonance approximation: field modes (incl. laser mode) near the resonance frequency  $\omega_A$ , hence only a narrow spectral band needed, with wavelength  $\lambda_A = 2\pi c/\omega_A \gg a_0$  (Bohr radius: size of atom)

“Rotating-wave approximation”: only resonant (energy-conserving) terms in the interaction Hamiltonian,  $a_k\sigma^\dagger$  and  $a_k^\dagger\sigma$ : one photon disappears and atom becomes excited (absorption) or the inverse (photon emission).

## 0.2 Density operators

Generalize “pure states” of ordinary quantum mechanics and combines them with “statistical mixtures” known from statistical physics.

A state vector  $|\psi\rangle$  corresponds to a density operator in the form of a projector  $\rho = |\psi\rangle\langle\psi|$ , this is what we are calling a “pure state”. In mathematical physics, this is the consistent definition of a quantum state since the unobservable phase of the vector  $|\psi\rangle$  disappears. (“States are rays in Hilbert space.”)

## Properties

**Definition: Density operator.**

linear operator on the Hilbert space  $\hat{\rho} : \mathcal{H} \mapsto \mathcal{H}$

$\hat{\rho}$  is hermitean (could be called an “observable”)<sup>1</sup>

$\hat{\rho}$  is positive:  $\langle \phi | \hat{\rho} | \phi \rangle \geq 0$ . This provides the probability interpretation; for a pure state,  $\langle \phi | \hat{\rho} | \phi \rangle = |\langle \phi | \psi \rangle|^2 \geq 0$ .

normalized (trace-class operator)  $\text{tr } \hat{\rho} = 1$

This definition also allows for “non-pure” or “mixed” states, for example

$$\hat{\rho} = p_1 |\psi_1\rangle \langle \psi_1| + p_2 |\psi_2\rangle \langle \psi_2|, \quad |\langle \psi_1 | \psi_2 \rangle| < 1 \quad (6)$$

with two pure states. The inequality ensures that the two states are not identical. But they are not necessarily orthogonal. If they are normalized, the sum of the two prefactors is  $p_1 + p_2 = 1$ . For this reason, one calls the mixed state (6) a “convex sum” which is a special linear superposition. Quite generally, the properties of density operators are preserved under convex sums (we could talk of a “convex vector space”)

$$\hat{\rho} = p_1 \hat{\rho}_1 + p_2 \hat{\rho}_2 \quad (7)$$

The degree of being pure can be quantified with two numbers,

$$\begin{aligned} \text{the purity } \text{Pu}(\hat{\rho}) = \text{tr}(\hat{\rho}^2 - \hat{\rho}) &= 0 \quad \text{if } \hat{\rho} \text{ is pure} \\ &< 0 \quad \text{mixed state} \end{aligned}$$

the von Neumann entropy  $S = -\text{tr}(\hat{\rho} \log \hat{\rho})$  which is the Shannon information of the set of eigenvalues  $\{p_1, \dots, p_N\}$  of  $\hat{\rho}$ . Indeed, these eigenvalues can be interpreted as a probability distribution.

## Quantum states of a two-level system (“qbit”)

Stationary states with respect to  $H_A$  (“free atom Hamiltonian”) are  $|e\rangle$  and  $|g\rangle$ . A general pure state is

$$|\psi(t)\rangle = \alpha(t)|e\rangle + \beta(t)|g\rangle \quad (8)$$

normalization  $|\alpha(t)|^2 + |\beta(t)|^2 = 1$  consistent with probability interpretation.

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<sup>1</sup>Indeed, if we are allowed for an infinite number of copies of the system prepared in  $\hat{\rho}$ , then the probabilities of finding the system in any of its states can be determined with arbitrary precision.

time evolution under  $H_A$  in the Schrödinger picture

$$\alpha(t) = \alpha(0) e^{-i\omega_A t/2} \quad (9)$$

$$\beta(t) = \beta(0) e^{i\omega_A t/2} \quad (10)$$

with our choice of zero energy. (Other choices: common, time-dependent phase factor.)

density matrix for a pure state  $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$  (projector), a hermitean  $2 \times 2$ -matrix

quantum average of atom observable  $\mathcal{A} = \sigma, \sigma_3, \dots$ :

$$\langle\mathcal{A}\rangle = \langle\psi|\mathcal{A}|\psi\rangle = \text{tr}[\mathcal{A}\rho] = \text{tr}[\rho\mathcal{A}] \quad (11)$$

mixed state: density matrix  $\rho$ , cannot be written as projector.

example: thermal state at temperature  $T$ ,

$$\rho = Z^{-1} (|g\rangle\langle g| + e^{-\hbar\omega_A/k_B T} |e\rangle\langle e|), \quad Z = 1 + e^{-\hbar\omega_A/k_B T} \quad (12)$$

Boltzmann factor for weighting the stationary states: combination of quantum and classical (ensemble) average.

geometric significance: Bloch vector and Bloch sphere

three components  $\mathbf{s} = (s_1, s_2, s_3)$ , expectation value of the spin operator

$$s_1 = \langle\sigma_1\rangle = \langle\sigma + \sigma^\dagger\rangle = 2 \text{Re}(\alpha^* \beta) \quad (13)$$

$$s_2 = \langle\sigma_2\rangle = (\cdot) \text{Im}(\alpha^* \beta) \quad (14)$$

$$s_3 = \langle\sigma_3\rangle = |\alpha|^2 - |\beta|^2 \quad (15)$$

sometimes complex notation used,  $s = \langle\sigma\rangle = (s_1 - i s_2)/2 = \alpha^* \beta$

pure state:  $1 = \mathbf{s}^2 = s_1^2 + s_2^2 + s_3^2 = 4|s|^2 + s_3^2$ , on the surface of the Bloch sphere.

“north pole”: excited state  $|e\rangle$  (consistent with “spin up”)

“south pole”: ground state  $|g\rangle$  (“spin down”)

on the equator: superposition states with equal weight, e.g.  $(|g\rangle + e^{i\phi}|e\rangle)/\sqrt{2}$ . Relative phase  $\phi$  determines position on the equator:

$$s = \pm \frac{1}{2} \frac{|g\rangle \pm |e\rangle}{\sqrt{2}}$$

$$s = \pm \frac{i}{2} \frac{|g\rangle \pm i|e\rangle}{\sqrt{2}} \quad (16)$$

(free) time evolution: rotation of the Bloch vector around the 3-axis with angular frequency  $\omega_A$ .

spin precession in the Heisenberg picture:

$$\frac{d\boldsymbol{\sigma}}{dt} = \frac{i}{\hbar} [H_A, \boldsymbol{\sigma}] = (\cdot)\omega_A \mathbf{e}_3 \times \boldsymbol{\sigma} \quad (17)$$

take expectation value: gives equation of motion for the Bloch vector.

rotation of the Bloch vector: pure states remain pure

mixed states: are located inside the Bloch sphere. For the thermal state (12), for example:

$$\mathbf{s} = -\mathbf{e}_3 \tanh(\hbar\omega_A/2k_B T) \quad (18)$$

in the “infinite temperature limit”,  $\mathbf{s} \rightarrow \mathbf{0}$ , the “completely mixed state”.

Quantitative measures of “being pure”: purity and entropy, see exercises.

### 0.3 Coherent + dissipative dynamics: Bloch equations

In this lecture, we are going to analyze the dynamics of a two-level system whose excited state  $|e\rangle$  can decay spontaneously with a rate  $\gamma$ . The equation of motion of the density operator  $\hat{\rho}(t)$  is then no longer given by the von Neumann equation, but contains an extra term

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar} [\hat{H}_A + \hat{H}_{AL}, \hat{\rho}] + \mathcal{L}(\hat{\rho}) \quad (19)$$

with the dissipative “superoperator”

$$\mathcal{L}(\hat{\rho}) = \gamma\sigma\hat{\rho}\sigma^\dagger - \frac{\gamma}{2} \underbrace{\{\sigma^\dagger\sigma, \hat{\rho}\}}_{\sigma^\dagger\sigma\hat{\rho} + \hat{\rho}\sigma^\dagger\sigma} \quad (20)$$

where the first term is related to a quantum jump  $|\psi\rangle \mapsto |g\rangle$  to the ground state. This can be interpreted as the result of detecting an emitted photon.

We observe that the trace of the density operator is conserved under Eq.(19):

$$\frac{d}{dt}\text{tr}\hat{\rho}(t) = 0; \quad (21)$$

this is consistent with the probability interpretation where the trace of  $\hat{\rho}$  is the probability of finding the system in “any state”. Note that sometimes, one deals with “lossy systems” where this property does not hold – the system can then “disappear” from a set of “detectable states”.

Here is the master equation for a two-level atom driven by a monochromatic laser field with Rabi frequency  $\Omega$  and coupled to the electromagnetic vacuum field: we use of course the frame rotating at the laser frequency  $\omega$ . It is simple to show that the corresponding unitary transformation leaves the master equation unchanged: only products  $\sigma_+\sigma_-$  appear, and the phase factors  $\sigma_{\pm} \mapsto \sigma_{\pm} e^{\pm i\omega t}$  cancel. The atomic Hamiltonian is thus given by the expression (1.31) of chapter 1 (lecture QO I)

$$H_A = -\frac{\hbar\Delta}{2} + \frac{\hbar\Omega}{2} (\sigma_+ + \sigma_-) \quad (22)$$

The detuning  $\Delta = \omega - \omega_A$  contains, of course, the renormalized transition frequency. The Rabi frequency  $\Omega$  can be chosen real.

The full master equation, written in terms of the standard representation (not the interaction representation), contains also a commutator

$$\frac{d\rho_A}{dt} = -\frac{i}{\hbar} [H_A, \rho_A] + \mathcal{L}[\rho_A] \quad (23)$$

The elements of the density matrix thus evolve according to

$$\frac{d\rho_{ee}}{dt} = i\frac{\Omega}{2} (\rho_{ge} - \rho_{eg}) - \gamma_e \rho_{ee} \quad (24)$$

$$\frac{d\rho_{eg}}{dt} = i\Delta \rho_{eg} + i\frac{\Omega}{2} (\rho_{gg} - \rho_{ee}) - \frac{\gamma_e}{2} \rho_{eg} \quad (25)$$

$$\frac{d\rho_{gg}}{dt} = -i\frac{\Omega}{2} (\rho_{ge} - \rho_{eg}) + \gamma_e \rho_{ee} \quad (26)$$

We thus learn that, starting from the ground state, the laser field first creates the coherence  $\rho_{eg}$  between the ground and excited states — and this only if there is a population difference. The coherence is then turned into an excited state population — if it contains a nonzero imaginary part. These equations give the basic description for many physical phenomena: absorption and emission of light, lasing, resonance fluorescence etc.