**Einführung in die Quantenoptik II** Sommersemester 2012

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**Problem Set 2** Hand out: Wed 17 Apr 13 hand in: Tue 30 Apr 13

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**Problem 2.1** – One photon, two modes (10 points)

In the lecture, we have studied a cavity mode which is a standing wave between the cavity mirrors. What is the momentum of the corresponding photons? This is not an easy question.

(i) Look up a formula by Louis de Broglie on the link between momentum and wavelength.

(ii) Look up a formula from electrodynamics for the momentum (density) in classical electrodynamics.

(iii) Check that a standing mode function is not an eigenfunction of the usual momentum operator (only 1D for simplicity, no polarisation)

$$\hat{p}_z \sin kz \neq p \sin kz \tag{2.1}$$

but it can be written as a sum of two eigenfunctions. Construct these two momentum eigenfunctions such that they are orthogonal with respect to the scalar product

$$(f, g) = \int_0^L \mathrm{d}z\, f^*(z) g(z) \tag{2.2}$$

(iv) We have seen in the lecture that the electric field operator can be written in the form (ignoring polarization again)

$$\hat{E}(z) = \mathcal{E}_{1\mathrm{ph}} \hat{a} \, \frac{\sin kz}{\sqrt{L/2}} + \mathrm{h.c.} \tag{2.3}$$

where $\mathcal{E}_{1\mathrm{ph}}$ is a suitable scale factor and the annihilation operator $\hat{a}$ is normalized such that $[\hat{a}, \hat{a}^\dagger] = 1$. Define an alternative set of operators $\hat{a}_+$ and $\hat{a}_-$ by the construction

$$\hat{a}_\pm \sim (f_\pm, \hat{E}^{(+)}) \tag{2.4}$$

where $\hat{E}^{(+)}$ is the first term in the sum (2.3), involving $\hat{a}$ and the $f_\pm$ are eigenfunctions of the momentum operator. Fix the unknown prefactor by imposing $[\hat{a}_+, \hat{a}_+^\dagger] = 1$ and check that one can write

$$\hat{a} = \alpha \hat{a}_+ + \beta \hat{a}_- \tag{2.5}$$

with suitable coefficients $\alpha$ and $\beta$.

(v) Discuss the state $\hat{a}^\dagger|0,0\rangle$ in the basis spanned by the number states $|n_+, n_-\rangle$ with $n_\pm$ photons created by the operators $\hat{a}_\pm$. How many photons does it contain?

[Bonus:] Use the electric and magnetic field operators to find an operator expression for the momentum of the cavity field. Consider a linear polarization for simplicity. **Hint**: use the space integral of the momentum density.

**Problem 2.2** – Field per photon (5 points)

In the lecture, we have found an expression similar to Eq.(2.3) for the electric field operator in a cavity:

$$\hat{\mathbf{E}}(\mathbf{x}) = \sqrt{\frac{\hbar\omega_c}{\mathcal{N}\varepsilon_0}}\mathbf{f}(\mathbf{x})\,(\hat{a} + \text{h.c.}) \tag{2.6}$$

(i) Fix the normalization factor $\mathcal{N}$ by requiring that the electric field energy in the empty cavity (zero photons) is equal to one half the zero-point energy $E_0 = \frac{1}{2}\hbar\omega_c$.

(ii) Look up typical numbers for the mode functions $\mathbf{f}(\mathbf{x})$ in a Fabry-Pérot cavity for visible light and give an estimate for the 'electric field per photon', as evaluated from

$$\langle\hat{\mathbf{E}}(\mathbf{x})\rangle\,, \qquad \langle\hat{\mathbf{E}}(\mathbf{x})^2\rangle \tag{2.7}$$

Compare this 'vacuum field' to other fields. One typical field in atomic physics is the 'saturation field' which is in order of magnitude

$$E_{\text{sat}} = \frac{\hbar/\tau}{ea_0} \tag{2.8}$$

where $\tau \sim 30\,\text{ns}$ is a typical lifetime of an excited atomic state, $e$ is the electron charge and $a_0$ the Bohr radius.

Can you find a strategy to make the vacuum field larger? (Keyword: 'micro-cavity'.)

**Problem 2.3** – Coherent states (5 points)

In the quantum mechanics lecture, you have seen coherent (or: Glauber) states, and they play an important role in quantum optics. With a complex number $\alpha$ and a boson operator $a$, they are defined by

$$|\alpha\rangle = \mathcal{N}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{2.9}$$

where $|n\rangle$ are the number states. Fix the normalization factor $\mathcal{N}$.

(i) Show that coherent states are eigenstates of the annihilation operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{2.10}$$

They are *not* eigenstates of $a^\dagger$.

(ii) Show that the 'photon statistics' (= probability to find $n$ photons) for a coherent state is

$$p_n = \frac{|\alpha|^{2n}\,\mathrm{e}^{-|\alpha|^2}}{n!} \tag{2.11}$$

and make a plot for $\alpha = 24 + 7\,\mathrm{i}$. The average photon number is $\langle n\rangle = |\alpha|^2$ and its variance $\Delta n^2 = |\alpha|^2$.

(iii) Show that two coherent states are not orthogonal:

$$\langle\alpha|\beta\rangle = \mathrm{e}^{-|\alpha-\beta|^2/2+\mathrm{i}\varphi} \tag{2.12}$$

with a phase $\varphi \sim \mathrm{Im}(\alpha^*\beta)$. One says that coherent states form an 'overcomplete basis'.

[Bonus:] There are no (normalizable) eigenstates of $a^\dagger$.