Einführung in die Quantenoptik II Sommersemester 2012

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Problem Set 4 Hand out: Wed 15 May 13 hand in: Wed 29 May 13

Problem 4.1 – Rate equations (20 points)

In quantum optics, one often deals with so-called two-level systems where only two stationary states are relevant: the ground state $|\mathrm{g}\rangle$ and the excited state $|\mathrm{e}\rangle$. In matrix calculations, we shall make the identification that the basis vectors $(1, 0)^T$ and $(0, 1)^T$ represent the excited and the ground state, respectively.

(i) Take an arbitrary superposition state $|\psi\rangle = \alpha|\mathrm{g}\rangle + \beta|\mathrm{e}\rangle$ and construct the corresponding density operator $\rho = |\psi\rangle\langle\psi|$. Give a matrix representation for ρ. What is the size of this matrix? What are its eigenvectors?

(ii) The diagonal elements $p_\mathrm{e} = \rho_\mathrm{ee}$ and $p_\mathrm{g} = \rho_\mathrm{gg}$ are called the populations. Calculate $p_\mathrm{g} + p_\mathrm{e}$ in the example of (i) and argue why this result must apply for any density operator (even non-pure ones that do not arise from a single superposition state).

(iii) Consider the set of rate equations

$$\frac{\mathrm{d}p_\mathrm{e}}{\mathrm{d}t} = -\Gamma p_\mathrm{e} + \gamma p_\mathrm{g}\,, \qquad \frac{\mathrm{d}p_\mathrm{g}}{\mathrm{d}t} = -\gamma p_\mathrm{g} + \Gamma p_\mathrm{e} \tag{4.1}$$

Give an interpretation for the coefficients γ and Γ (units?). Show that $p_\mathrm{g} + p_\mathrm{e}$ is a constant. Calculate the stationary populations and argue that γ/Γ plays the role of a Boltzmann factor.

(iv) The rate equations cannot be generated by a Hamiltonian acting on the two-level system alone. To show this, consider the so-called purity

$$P = \frac{\mathrm{tr}\,\rho^2}{\mathrm{tr}\,\rho} \tag{4.2}$$

Show that $\mathrm{d}P/\mathrm{d}t = 0$ when ρ evolves according to the Schrödinger (& von Neumann) equation $\mathrm{i}\hbar\,\mathrm{d}\rho/\mathrm{d}t = [H, \rho]$. Calculate $\mathrm{d}P/\mathrm{d}t$ from the rate equations (4.1) for the restricted class of density operators that are diagonal $\rho = p_\mathrm{e}|\mathrm{e}\rangle\langle\mathrm{e}| + p_\mathrm{g}|\mathrm{g}\rangle\langle\mathrm{g}|$.

(v) Solve the rate equations (4.1) analytically for a two-level system that is initially in the excited state, i.e., $p_\mathrm{e}(0) = 1$, $p_\mathrm{g}(0) = 0$. Plot the purity P and the von Neumann entropy

$$S = -p_\mathrm{e}\log p_\mathrm{e} - p_\mathrm{g}\log p_\mathrm{g} \tag{4.3}$$

as a function of time. Speculate about the interpretation of your result.