Einführung in die Quantenoptik II Sommersemester 2012

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Problem Set 6 Hand out: Wed 19 Jun 13 hand in: Wed 03 Jul 13

Problem 6.1 – Jaynes-Cummings-Paul model (12 points)

A two-level system that is coupled to just one cavity mode is the subject of the so-called Jaynes-Cummings-Paul model. This system is described by the Hamiltonian in the resonance approximation (notation of the lecture for the ladder operators)

$$H_{\rm JCP} = \hbar\omega_c a^\dagger a + \hbar\omega_A \sigma^\dagger\sigma + \hbar(g a^\dagger \sigma + g^* \sigma^\dagger a) \tag{6.1}$$

(i) Explain in words the action of the interaction Hamiltonian and the meaning of the resonance approximation.

(ii) Show that the JCP Hamiltonian commutes with the "excitation" operator

$$N = a^\dagger a + \sigma^\dagger \sigma \tag{6.2}$$

Argue that the eigenvalues of N are the integers $n = 0, 1, 2\ldots$ and that each nonzero eigenvalue is doubly degenerate. Write down two basis vectors. We call in the following $\mathcal{H}^{(n)}$ the sub-space spanned by the eigenvectors with eigenvalue $n \neq 0$ to N.

(iii) Show that the Hamiltonian is represented by the following 2×2 matrix in the subspace $\mathcal{H}^{(n)}$ (explain the choice of the basis vectors):

$$H^{(n)} = \hbar \begin{pmatrix} (n-1)\omega_c + \omega_A & g^*\sqrt{n} \\ g\sqrt{n} & n\omega_c \end{pmatrix} \tag{6.3}$$

and that the state $|0, g\rangle$ is a singly degenerate eigenstate of $H_{\rm JCP}$. Make a sketch of the eigenvalues of $H_{\rm JCP}$.

(iv) Show that the following *Ansatz* yields eigenstates of $H^{(n)}$:

$$\begin{pmatrix} {\rm e}^{-{\rm i}\varphi}\cos\theta \\ {\rm e}^{{\rm i}\varphi}\sin\theta \end{pmatrix}, \quad \begin{pmatrix} {\rm e}^{-{\rm i}\varphi}\sin\theta \\ -{\rm e}^{{\rm i}\varphi}\cos\theta \end{pmatrix} \tag{6.4}$$

fix the angles φ, θ, and write the state $(0, 1)^T$ as a linear combination of these eigenstates.

Problem 6.2 – Two-level polarizability (8 points)

In the lecture, we have found the following expression for the polarizability of a two-level system (here, ω can be interpreted as the laser frequency and $\mathbf{d}_{ge}$ is taken real)

$$\alpha(\omega) = \frac{\mathbf{d}_{ge}\mathbf{d}_{ge}\cdot}{\hbar}\left(\frac{\mathrm{i}}{\gamma/2 - \mathrm{i}(\omega - \omega_A)} - \frac{\mathrm{i}}{\gamma/2 - \mathrm{i}(\omega + \omega_A)}\right)(\rho_{gg} - \rho_{ee}) \tag{6.5}$$

(i) Check the signs by verifying that $\alpha(-\omega) = [\alpha(\omega)]^*$.

(ii) Make a plot of $\operatorname{Im}\alpha(\omega)$ and $\omega \operatorname{Im}\alpha(\omega)$ for a fixed value of the inversion $\rho_{gg} - \rho_{ee} > 0$. What would you get for an inverted system?

(iii) Do the symmetry properties of (i) and (ii) survive if the stationary value for the inversion

$$\rho_{gg} - \rho_{ee} = \frac{\gamma^2/4 + \Delta^2}{|\Omega|^2/2 + \gamma^2/4 + \Delta^2} \tag{6.6}$$

is employed in Eq.(6.5)? (Rabi frequency Ω, detuning $\Delta = \omega - \omega_A$)