Einführung in die Quantenoptik II

Sommersemester 2014
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Übungsaufgaben Blatt 2

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Hinweis. Die Übungsaufgaben sind ein Versuch, verschiedene 'Geschmäcker' zu bedienen: mal geht es um Abschätzungen, Einheiten, Größenordnungen. Mal gibt es einiges zu rechnen. Häufig ist schon das Interpretieren des Aufgabentextes Teil der Herausforderung.

Es gilt die Regel: Lassen Sie sich von Fehlern in den angegebenen Formeln nicht verwirren. Im Zweifelsfall fehlt eben im Aufgabentext ein Faktor 2, π, i, -1 ...

Problem 2.1 – Bath correlations (12 points)

In the lecture, we have introduced the basic elements of a bath: an infinite collection of harmonic oscillators with frequencies ω_k and annihilation (creation) operators a_k ($a_k^\dagger$). Consider the following 'bath variable' (or collective operator):

$$B = \sum_k \left(g_k a_k + g_k^* a_k^\dagger \right) \tag{2.1}$$

(1) Is B hermitean? Is B a bounded operator?
(2) Compute the time-dependent operator $B(t)$ in the Heisenberg picture such that $B(0) = B$.
(3) Calculate for a bath in the ground state (subscript 0) the autocorrelation function

$$C_0(t - t') = \langle B(t) B(t') \rangle_0 \tag{2.2}$$

Is this definition correct with respect to a common shift of the times t and t'? (Try to repeat the calculation with $t + s$ and $t' + s$).
(4) Show that the spectral density

$$J_0(\omega) = 2\pi \sum_k |g_k|^2 \delta(\omega - \omega_k) \tag{2.3}$$

is the Fourier transform of $C_0(t)$.
(5) [5 bonus points] Repeat the calculation for a bath in thermal equilibrium at temperature $k_B T = \hbar/\beta$. The spectral density $J_T(\omega)$ then becomes temperature dependent. Remember that

$$\langle a_k^\dagger a_l \rangle_T = \frac{\delta_{kl}}{\mathrm{e}^{\beta\omega_k} - 1} \tag{2.4}$$

Observe that the spectrum $J_T(\omega)$ is qualitatively different for low frequencies $\hbar\omega_k \ll k_B T$ and high frequencies $\hbar\omega_k \gg k_B T$. From this, one can estimate a correlation time $\tau_c \sim \beta$ for a thermal bath. Calculate this time at room temperature.

Problem 2.2 – Maximal dipole moment (8 points)

In the lecture, we have seen the formula for the dipole moment of a two-level atom (in the stationary state, pumping rate λ_e into the excited state, with an electric field $\mathbf{E}$ at frequency $\omega_L = \omega_A + \Delta$):

$$\langle \mathbf{d}(t) \rangle = \mathbf{d}_{ge} \frac{\mathbf{d}_{ge} \cdot \mathbf{E}/\hbar}{\Delta + \mathrm{i}\Gamma} \frac{\lambda_e (1/\gamma - 1/\gamma_g)(\Delta^2 + \Gamma^2)}{\Delta^2 + \Gamma^2 + 2(\Gamma/\gamma)|\mathbf{d}_{ge} \cdot \mathbf{E}|^2/\hbar^2} \mathrm{e}^{-\mathrm{i}\omega_L t} + \mathrm{c.c.} \tag{2.5}$$

(1) Check the units of this formula.

(2) Find the maximum amplitude $\langle \mathbf{d} \rangle_{\max}$ for the dipole moment, as the field amplitude $\mathbf{E}$ is varied ('saturation' at high intensity). How large would be the displacement of an electron for this dipole? (Typical numbers: λ_e, Δ, Γ, γ all of the same order.)

(3) Compare the maximum medium polarization $\mathbf{P}_{\max} = N\langle \mathbf{d} \rangle_{\max}$ (N is the particle density) for a typical medium to other polarizations. (Perhaps the keyword piezoelectricity is helpful here.)