Einführung in die Quantenoptik II

Sommersemester 2014
Carsten Henkel

Übungsaufgaben Blatt 3 Ausgabe: 30. April 2014
Abgabe: 16. Mai 2014

Problem 3.1 – Jaynes-Cummings-Paul model (8 points)

The simplest model for a laser is built from one two-level atom and one field mode. This can be described by the so-called Jaynes–Cummings–Paul Hamiltonian

$$H_{\rm JCP} = \hbar\omega_A \sigma^\dagger\sigma + \hbar\omega_c a^\dagger a - \mathbf{d}\cdot\mathbf{E}(\mathbf{r}_A) \tag{3.1}$$

(a) Write down the interaction Hamiltonian V using the atomic dipole operators σ, $\sigma^\dagger$ and the photon operators a, $a^\dagger$. Evaluate its matrix elements $\langle e, 1|V|g, 0\rangle$ and $\langle g, 1|V|e, 0\rangle =: \hbar g$. (No confusion between the atomic ground state $|g\rangle$ and the coupling constant g.)

(b) Justify in words why the resonance (or 'rotating-wave') approximation is often appropriate in quantum optics. Remember that in this approximation, one works with the simplified interaction

$$-\mathbf{d}\cdot\mathbf{E}(\mathbf{r}_A) \approx V_{\rm JCP} = \hbar\left(g a^\dagger\sigma + g^*\sigma^\dagger a\right) \tag{3.2}$$

Show that its only non-zero matrix elements are $\langle g, n{+}1|V_{\rm JCP}|e, n\rangle = \hbar g\sqrt{n+1}$, and conclude that the JCP Hamiltonian can be diagonalized exactly in the subspaces spanned by $|e, n-1\rangle$ and $|g, n\rangle$ (for $n = 1, 2\ldots$).

(c) Consider the simple case of an initial state with exactly n photons and the atom in the excited state, and assume perfect resonance $\omega_A = \omega_c$. Show that Rabi oscillations occur with a frequency $g\sqrt{n+1}$. ($n = 0$: 'vacuum Rabi oscillations'.)

(d) In a bright coherent state, the photon number is distributed around a mean value $\bar{n}$ with a typical width $\Delta n = \bar{n}^{1/2} \gg 1$. Argue that the amplitude of the Rabi oscillations decays because the different photon numbers get out of phase and justify the estimation $|g|\tau \sim 1$ for the typical decay time τ.

This decay is called 'collapse' of Rabi oscillations. The envelope of the oscillations does not follow an exponential, but approximately a gaussian if $\bar{n}$ is large enough. At even larger times $|g|t \sim \bar{n}^{1/2}$, the Rabi oscillations 'revive' ('collapse and revival').

Problem 3.2 – Scully-Lamb laser theory (6 points)

In the lecture, we have seen the derivation of the Scully-Lamb master equation for the photon statistics

$$\frac{\mathrm{d}p_n}{\mathrm{d}t} = -n\kappa\, p_n + (n+1)\kappa\, p_{n+1} - (n+1)G(n)\, p_n + nG(n-1)\, p_{n-1} \tag{3.3}$$

where the dependence of the gain $G(n)$ on the photon number is a model for saturation. (a) Show that the recurrence relations ('detailed balance')

$$p_{n+1} = p_n \frac{G(n)}{\kappa} \tag{3.4}$$

lead to a stationary solution to the master equation.

(b) Consider the simple gain model $G(n) = G_0/(1 + Bn)$. Show that the laser threshold is given by the condition $G_0 = \kappa$ and that for $G_0 > \kappa$, the photon number with the maximum probability is approximately

$$n_{\max} \approx \frac{1}{B}\left(\frac{G_0}{\kappa} - 1\right). \tag{3.5}$$

Problem 3.3 – Lindblad master equation for gain saturation (6 points)

A technically difficult problem is to include gain saturation into the master equation for a laser. A simple approach is to take as the Lindblad operator for gain

$$G = \sqrt{G_0}\, a^\dagger(1 - \beta a^\dagger a) \tag{3.6}$$

Calculate the master motion for the photon statistics $p_n(t) = \langle n|\rho(t)|n\rangle$ from this gain operator and the usual linear loss operator $L = \sqrt{\kappa}\, a$:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[H, \rho] + L\rho L^\dagger - \frac{1}{2}\left\{L^\dagger L\,, \rho\right\} + G\rho G^\dagger - \frac{1}{2}\left\{G^\dagger G\,, \rho\right\} \tag{3.7}$$

You may assume for simplicity that $H = \hbar\omega_c a^\dagger a$. Compare to the Scully-Lamb laser theory of the lecture [Eq.(3.3)] where gain saturation is taken in the form $G(n) = G_0/(1 + Bn)$.