Einführung in die Quantenoptik II

Sommersemester 2014
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Problem 4.1 – Positive maps and the Kraus representation (10 points)

In the lecture, we have encountered dynamical (or completely positive = CP) maps. Recall that such a map $\Lambda : \rho \mapsto \Lambda(\rho)$

- is linear: $\Lambda(p\rho + q\rho') = p\Lambda(\rho) + q\Lambda(\rho')$ for real $p, q \geq 0$
- is positive and preserves the trace: $\langle\psi|\Lambda(\rho)|\psi\rangle \geq 0$ for all $|\psi\rangle$ and all ρ, $\operatorname{tr}\Lambda(\rho) = \operatorname{tr}\rho$.
- is completely positive: if Λ is extended to $\Lambda \otimes \mathbb{1}$ on a larger Hilbert space, then this map is positive.

(1) Show that unitary time evolution for a closed system, $\rho \mapsto U\rho U^\dagger$ with unitary U, is a CP map.
(2) Show that CP maps form a convex set: $p\Lambda + q\Lambda'$ is CP for real $0 \leq p, q \leq 1$, $p + q = 1$ if Λ and Λ' are CP.
(3) Conclude that the 'Kraus operation' defined as follows is a CP map:

$$\rho \mapsto \sum_k p_k U_k \rho U_k^\dagger\,, \quad \sum_k p_k = 1\,, \quad U_k \text{ unitary} \tag{4.1}$$

Problem 4.2 – Spectra – Fourier vs Laplace (10 points)

Correlation functions $\langle B(t)A(t')\rangle$ are nearly always calculated under stationary conditions, i.e., $\langle B(t+T)A(t'+T)\rangle = \langle B(t)A(t')\rangle$ for any time shift T.
(1) Show that stationary correlations only depend on the time difference $\tau = t - t'$.
(2) Show that if A and B are hermitean conjugates of each other, $A^\dagger = B$, then

$$\int_{-\infty}^{+\infty} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega\tau} \langle A^\dagger(t+\tau)A(t)\rangle = 2\,\mathrm{Re} \int_{0}^{+\infty} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega\tau} \langle A^\dagger(t+\tau)A(t)\rangle \tag{4.2}$$

so that one only needs 'time-ordered (stationary) correlations'.
(3) Assume that the correlation function becomes a sum of damped exponentials (complex λ_k with $\mathrm{Re}\,\lambda_k \geq 0$ are possible)

$$\langle A^\dagger(t+\tau)A(t)\rangle = \sum_k \mathrm{e}^{-\lambda_k \tau} S_k \tag{4.3}$$

and show that the half-sided Fourier integral in Eq.(4.2) yields a spectrum which is a sum of Lorentzians. What happens when $\operatorname{Re}\lambda_k \to 0$?

Problem 4.3 – Positive maps and transposition (10 points)

[Suggested reading for the interested student.] Transposition is an example of a positive, but not completely positive map. To see this, consider a two-level system and define the transposed density matrix in a given basis $|a\rangle, a = e, g$, by

$$\rho = \sum_{a,b} \rho_{ab} |a\rangle\langle b| \mapsto \rho^{\mathrm{T}} = \sum_{a,b} \rho_{ab} |b\rangle\langle a| \tag{4.4}$$

(1) Show that the matrix representing ρ^{T} is the transpose of ρ; its matrix elements are the complex conjugates of those of ρ: $(\rho^{\mathrm{T}})_{ab} = (\rho_{ab})^*$. Show that ρ^{T} is again a density operator.

(2) For a system of two qubits, take the basis $\{|ee\rangle, |ge\rangle, |eg\rangle, |gg\rangle\}$ and consider in this basis a two-qubit density matrix (greek capital ρ)

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4.5}$$

with 2×2 block matrices A, B, C, D. Check that $A^\dagger = A$, $D^\dagger = D$, $B^\dagger = C$, $\operatorname{tr}(A + D) = 1$. Calculate the reduced density matrices $\operatorname{tr}_{2,1}(P)$ for the first and the second qubit (the index at the trace means: 'trace out this sub-system')

$$\operatorname{tr}_2(P) = A + D\,, \qquad \operatorname{tr}_1(P) = \begin{pmatrix} \operatorname{tr} A & \operatorname{tr} B \\ \operatorname{tr} C & \operatorname{tr} D \end{pmatrix} \tag{4.6}$$

(3) The transposed density matrix is of course

$$P^{\mathrm{T}} = \begin{pmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix} \tag{4.7}$$

The *partial transpose* P^Γ ('a half of a T', applied to the *second* system) is the operator defined by

$$P = \sum_{a,b,c,d} P_{ac,bd} |a\,c\rangle\langle b\,d| \mapsto P^\Gamma = \sum_{a,b,c,d} P_{ac,bd} |a\,d\rangle\langle b\,c| \tag{4.8}$$

Show that for two qubits, its matrix representation is

$$P^\Gamma = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \tag{4.9}$$

'This is intuitive': the reduced state for the second qubit, $\operatorname{tr}_1(P^\Gamma)$, is the transpose of $\operatorname{tr}_1(P)$, while nothing changes when we trace out the second qubit. The 'total transpose' of this is the partial transpose on the first system, which we denote $P^{\rceil}$

$$P^{\rceil} = (P^{\Gamma})^{\mathrm{T}} = \begin{pmatrix} A^{\mathrm{T}} & B^{\mathrm{T}} \\ C^{\mathrm{T}} & D^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} A^{*} & C^{*} \\ B^{*} & D^{*} \end{pmatrix} \tag{4.10}$$

(4) Show that all these maps are involutions: apply them twice, and nothing has changed.
(5) Consider the state (this one is called 'completely entangled', a projector on $|ge\rangle + |eg\rangle$)

$$P = \begin{pmatrix} 0 & & & \\ & 1 & 1 & \\ & 1 & 1 & \\ & & & 0 \end{pmatrix} \tag{4.11}$$

where non-written matrix elements are zero, and show that its partial transposes $P^{\Gamma} = P^{\rceil}$ are *not* positive.