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## Introduction to Quantum Optics II

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## Einführung in die Quantenoptik II

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Die Vorlesung Quantenoptik II setzt den ersten Teil des Kurses aus dem WS 2013/14 fort, kann aber auch unabhängig davon gehört werden. Die benötigten Grundbegriffe (Wechselwirkung Licht-Materie, Quantisierung des Strahlungsfelds) werden hier kurz wiederholt.

Ziel der Vorlesung ist Einführung in ein aktuelles Gebiet der Forschung an Hand von einigen Schlüssel-Experimenten zu Licht und Photonen. Im Vergleich zur Experimentalphysik wird etwas mehr Gewicht auf die theoretische Beschreibung und ihre physikalische Interpretation gelegt, die Details werden aber mit den Hörern abgestimmt. Im vorläufigen Programm sind um die Experimente herum verschiedene methodische Techniken eingeplant, die diverse Aspekte für die Beschreibung von Photonen, Lichtfeldern und ihrer Kopplung an Materie beleuchten. Hinweise zur Literatur und auf ein Skriptum sind auf dem web site der Quantenoptik in Potsdam zu finden:

[www.quantum.physik.uni-potsdam.de](http://www.quantum.physik.uni-potsdam.de)

1. Quantum Theory of Laser and Micro-Maser
  - active medium
  - semiclassical description
  - rate equations and photon statistics
  - dynamics in phase space and Schawlow-Townes line width
2. Resonance Fluorescence
  - physical interpretation of master (Bloch) equation
  - spectra, correlation functions, regression formula
  - Mollow triplet
  - intensity correlations and anti-bunching
3. Squeezing in a Cavity

- input-output formalism and quantum Langevin equation
- nonlinear medium and squeezing interaction
- spectra of squeezed quadrature operators

#### 4. Vacuum fluctuations

- the ‘wrongest formula in physics’
- observable consequences: spontaneous emission, Lamb shift, Casimir force

## 0.1 System+bath models

### Introduction

The concept of a system in a bath is a typical setting in quantum optics, but applies also to other areas of physics (condensed matter, statistical physics). The main idea is that we want to make a simple model where a system (Hamiltonian  $H_S$ ) is in contact with an “environment” whose detailed behaviour is not accessible. By definition, we cannot make measurements on the “bath”. It is therefore characterized by only a few parameters.

A simple case is the one of a “bosonic bath” where the total Hamiltonian takes the form

$$H = H_S + H_B + V \quad (1)$$

where the bath is described by a collection of bosons, i.e., harmonic oscillators

$$H_B = \sum_k \hbar \omega_k a_k^\dagger a_k = \sum_k \left( \frac{p_k^2}{2m_k} + \frac{m_k}{2} \omega_k^2 q_k^2 \right) + \text{const.} \quad (2)$$

with annihilation and creation operators  $a_k, a_k^\dagger$  or conjugate “position” and “momentum” coordinates  $q_k, p_k$ . The system-bath interaction  $V$  is a product of system and bath observables and can be written in the form

$$V = S^\dagger B + \text{h.c.} \quad (3)$$

Sometimes a coupling constant  $\lambda$  is introduced here, writing  $V = \lambda S^\dagger B + \text{h.c.}$ : this can be useful to count orders in perturbation theory. For the

moment, we have incorporated the coupling constant in the definition of the bath observable  $B$ . Typically, this is a ‘collective’ degree of freedom containing many bath modes

$$B = \sum_k g_k a_k (+\text{h.c.}) \quad (4)$$

This is often taken as a hermitean operator (hence the ‘+ h.c.’), but in some applications, it is not.

Example of a system “atom” coupled to the bath “electromagnetic field”:  $S$  is the electric dipole operator and  $B$  the electric field operator. In quantum optics, one frequently uses a non-hermitean bath operator  $B$ , expanded as a sum of single-mode operators  $a_k$ .

### Bath spectral density

Since not much information is available about the bath, it is characterized in a simple way, using a few numbers. Its temperature, some characteristic rate of energy exchange with the system (e.g., the rate of spontaneous emission, or a friction coefficient), some characteristic frequency for its spectral features. The bath is described by a continuous distribution of frequencies, i.e., eventually one goes from a summation over  $k$  into a frequency integral. This is encoded in the bath spectral density defined by (the factor  $2\pi$  is a convention not used by everybody)

$$J(\omega) = 2\pi \sum_k |g_k|^2 \delta(\omega - \omega_k) \quad (5)$$

which is appropriate when the bath operator takes the form

$$B = \sum_k g_k a_k \quad (6)$$

Roughly speaking,  $J(\omega)$  encodes how many bath degrees of freedom have a frequency near  $\omega$ , weighted with the coupling strength (parameter  $g_k$ ) to the system. Technically, the spectral density  $J(\omega)$  is related by a Fourier transform to the following autocorrelation function of  $B$ , assuming that the bath is in the vacuum state

$$C(t - t') = \langle B(t) B^\dagger(t') \rangle \quad (7)$$

Here, the time-dependent operator  $B(t)$  is evolving in the so-called interaction picture – which simply means that we use the free time evolution of the operators  $a_k(t) = a_k(0) e^{-i\omega_k t}$  under the bath Hamiltonian  $H_B$ .

In the exercises, you learn more about the physical meaning of  $J(\omega)$  and the role of temperature.

### Reduced density operator

The information about the system is encoded in its so-called *reduced density operator*  $\rho$ . This is defined by

$$\rho = \begin{cases} \text{tr}_B |\Psi_{SB}\rangle\langle\Psi_{SB}| \\ \text{tr}_B \rho_{SB} \end{cases} \quad (8)$$

where the definition depends on how the total system is described. If its wave function (quantum state)  $|\Psi_{SB}\rangle$  is known, the first line applies. If its density operator  $\rho_{SB}$  is given, the second line. The operation “ $\text{tr}_B$ ” means “take the partial trace over the bath variables” or “trace out the bath”. It is defined in Eq.(11) below. In both cases, the system itself is *not* described by a pure state (a wave function), but a mixed one, and we have to use a density operator.

### Definition of the partial trace

Take a basis  $\{|n\rangle, n = \dots\}$  for the system Hilbert space and a basis  $\{|b\rangle, b = \dots\}$  for the bath Hilbert space. (At least the second basis is likely to be infinite-dimensional.) Calculate the elements of the density matrix by summing over the bath states

$$\rho_{mn} = \sum_b \langle m, b | \rho_{SB} | n, b \rangle \quad (9)$$

where the same bath state appears on both sides. This is why one speaks of a “partial trace”. Then construct the density operator, acting on the Hilbert space of the system

$$\rho = \sum_{mn} |m\rangle\langle n| \rho_{mn} \quad (10)$$

The operation “partial trace” or “tracing out the bath” is therefore given by the map

$$P \mapsto \rho = \text{tr}_B P = \sum_{mn,b} |m\rangle\langle m, b| P |n, b\rangle\langle n| \quad (11)$$

### System observables

An important statement is the following: *The reduced density operator encodes the expectation values for all system observables.* This is the “luxury” one typically has in quantum optics: all system observables can potentially be measured and have to be dealt with in the theory. In other fields of physics, one is happy with a reduced set of “relevant observables” (current and charge distribution, etc). In this context, it is often not necessary to calculate the full (system) density operator, a smaller set of “observables of interest” is sufficient.

The usual rule for expectation values

$$\langle S \rangle = \text{tr}(S\rho) = \text{tr}(\rho S) \quad (12)$$

where the trace is now over the system Hilbert space. This rule generalizes the expectation value for a pure state (full knowledge about the wave function) where  $\langle S \rangle = \langle \psi | S | \psi \rangle$ . Under the trace, one can permute cyclically, since the trace of  $\rho$  exists (“ $\rho$  is trace class”).

Now, be  $S$  any system observable. According to orthodox quantum mechanics, it is a linear operator acting on the system’s Hilbert space. Hence, we can write for the action on the basis state  $|n\rangle$ :

$$S|n\rangle = \sum_m |m\rangle S_{mn}, \quad S_{mn} = \langle m | S | n \rangle \quad (13)$$

(Note the position of the indices.) There is thus a one-to-one correspondence, in a given basis, between the matrix  $(S_{mn})$  and the operator  $S$ . The expectation value becomes, using (10) and (13):

$$\begin{aligned} \langle S \rangle &= \text{tr} \sum_{mn} (S|n\rangle\langle m| \rho_{nm}) \\ &= \text{tr} \sum_{mn'n} (|m'\rangle S_{m'n} \langle m| \rho_{nm}) \\ &= \sum_{kn} S_{kn} \rho_{nk} \end{aligned} \quad (14)$$

which is just the trace of the matrix product between the matrix representing  $S$  and the density matrix  $\rho$ . We have simply checked that the basis-free definition (12) is consistent with the rules for matrix algebra.

So, how does “any” system operator look like? It is given by any matrix  $S_{mn}$  that is compatible with a few basic requirements. Sometimes one restricts to self-adjoint operators  $S$ , hence hermitean matrices  $(S_{mn})$ , but this can be relaxed. We could also choose a matrix  $S_{mn}$  that has a single element one somewhere (at the indices  $mn$ , say) the others being zero. Then the corresponding expectation value gives simply the matrix element  $\rho_{nm}$  – the *elements of the density matrix are observable*.

### Programme for open quantum systems

We can formulate a few basic questions:

- the stationary state of the system  $\rho_{ss}$ , given the coupling to the bath at temperature  $T$  and some “control fields” (like a laser that illuminates an atom)
- the state of thermal equilibrium  $\rho_{eq}$ , given the coupling to the bath at temperature  $T$
- the dynamics (time evolution)  $\rho(t)$  of the system

To the second question, a formal answer can be given (P. Hänggi, Augsburg):

$$\rho_{eq}(T) = \text{tr}_B \left( \frac{1}{Z_{SB}} e^{-\beta \hat{H}} \right) \neq \frac{1}{Z_S} e^{-\beta \hat{H}_S} \quad (15)$$

where  $\beta = 1/k_B T$  is the inverse temperature, and the partial trace of the canonical equilibrium density operator for the total system is taken. Note that the two expressions differ because the system-bath coupling  $V$  enters the Hamiltonian  $H$  in the first one, while only the system Hamiltonian  $H_S$  appears in the second one. The two coincide only for a sufficiently weak system-bath coupling.

In general, the stationary density operator  $\rho_{ss}$  is equal to the equilibrium one only if the control fields are sufficiently weak. This means that we have to develop tools to handle *non-equilibrium systems*.

The answer to the third question, the time evolution of the density operator, is given by different methods (and approximations)

- master equations
- quantum Langevin equations
- path integrals with influence functionals
- quantum Monte Carlo techniques

a number of which are going to discuss in this lecture.