

Chapter 4

Vacuum fluctuations

4.1 Spontaneous decay

Dirac in his 1927 paper ‘The Quantum Theory of the Emission and Absorption of Radiation’ (Dirac, 1927) quantizes the electromagnetic field (with photon annihilation and creation operators). As a first application, he calculates the rate for the spontaneous decay of excited atomic states, also known as the ‘Einstein A-coefficient’.

The calculation is based on time-dependent perturbation theory and yields a rate given by ‘Fermi’s Golden Rule’:

$$\gamma_{e \rightarrow g} = \frac{2\pi}{\hbar} \sum_{\kappa} |\langle g, \kappa | V | e, \text{vac} \rangle|^2 \delta(\hbar\omega_{\kappa} - \hbar\omega_A) \quad (4.1)$$

Here, $|\kappa\rangle$ is a state with one photon in the mode with quantum numbers κ (in a plane-wave basis: wave vector and polarization). V is the atom-field interaction Hamiltonian. The δ -function ensures Bohr’s rule for the photon frequency ω_{κ} : it is given by the Bohr frequency ω_A of the two levels.

Calculation with a quantized light field: here with a discrete sum over wave vectors (in a ‘quantization volume’ Ω). The interaction Hamiltonian is the electric dipole coupling:

$$V = -(\mathbf{d}_{ge}\sigma + \mathbf{d}_{ge}^*\sigma^\dagger) \cdot \sum_{\kappa} \sqrt{\frac{\hbar\omega_{\kappa}}{2\varepsilon_0\Omega}} (\mathbf{e}_{\kappa} e^{i\mathbf{k}\cdot\mathbf{r}_A} \hat{a}_{\kappa} + \text{h.c.}) \quad (4.2)$$

and gives a squared matrix element

$$|\langle g, \kappa | V | e, \text{vac} \rangle|^2 = |\mathbf{e}_{\kappa}^* \cdot \mathbf{d}_{ge}|^2 \frac{\hbar\omega_{\kappa}}{2\varepsilon_0\Omega} \quad (4.3)$$

Summing over the two polarizations (label μ) gives for a real-valued transition dipole \mathbf{d}_{ge} :

$$\sum_{\mu} |\mathbf{e}_{\mathbf{k}\mu}^* \cdot \mathbf{d}_{ge}|^2 = |\mathbf{d}_{ge}|^2 - |\hat{\mathbf{k}} \cdot \mathbf{d}_{ge}|^2 = |\mathbf{d}_{ge}|^2 \sin^2 \theta \quad (4.4)$$

where θ is the angle between the photon wave vector and the dipole. Note that a classical dipole has the angular dependence of its radiation (emission pattern).

Exercise. For a circularly polarized transition dipole, one gets something like

$$\sum_{\mu} |\mathbf{e}_{\mathbf{k}\mu}^* \cdot \mathbf{d}_{ge}|^2 = |\mathbf{d}_{ge}|^2 (1 - \sin^2 \theta)$$

where now θ is the angle between the emission direction and the vector normal to the plane of circular polarization.

Integration over the wave vector: we go from a summation to an integral with the rule (for periodic boundary conditions):

$$\sum_{\mathbf{k}} = \Omega \int \frac{d^3 k}{(2\pi)^3} = \Omega \int \frac{k^2 dk \sin \theta d\theta d\phi}{(2\pi)^3} \quad (4.5)$$

using spherical coordinates. Note that the quantization volume Ω drops out.

Integrate over the direction of \mathbf{k} :

$$\int \sin \theta d\theta d\phi \sin^2 \theta = \frac{8\pi}{3} \quad (4.6)$$

The integral over the length dk is handled by the δ -function $\delta(\hbar\omega_{\kappa} - \hbar\omega_A) = 1/(\hbar c)\delta(k - \omega_A/c)$. The final result is

$$\gamma_{e \rightarrow g} = \frac{2\pi(\omega_A/c)^2}{\hbar^2 c} \frac{8\pi}{3} |\mathbf{d}_{ge}|^2 \frac{\hbar\omega_A}{2\varepsilon_0} = \frac{|\mathbf{d}_{ge}|^2 \omega_A^3}{3\pi\varepsilon_0 \hbar c^3} \quad (4.7)$$

Typical order of magnitude: lifetime $\tau_e = 1/\gamma_{e \rightarrow g} \approx 10$ ns, depending on the atomic level. The lifetime is shorter for hydrogen because its transition frequency ω_A is relatively high. The 10 ns figure is typical for the strong $p \rightarrow s$ transitions (D1 and D2 lines) in the alkali atoms. In highly excited states (called Rydberg states), it happens that although the transition dipole becomes large (the electron orbit has a large size), the transition frequencies are becoming so low (in the microwave band) that lifetimes in the order of a microsecond become possible.

4.2 Wigner-Weisskopf master equation

Motivation

Experimentally observed: excited states decay at some rate γ to the ground state. Evolution of population: *rate equation*

$$\begin{aligned}\dot{p}_e &= -\gamma p_e \\ \dot{p}_g &= \gamma p_e\end{aligned}\tag{4.8}$$

Total population conserved. Stationary state: $p_e = 0$.

Problem: not possible to obtain with a Hamiltonian. Standard interaction $V = \frac{1}{2}\hbar\Omega(|g\rangle\langle e| + |e\rangle\langle g|)$ couples population $p_e = \langle e|\rho|e\rangle = \rho_{ee}$ to an off-diagonal matrix element (a “coherence”)

$$\dot{p}_e = -\frac{i}{\hbar}\langle e|[V, \rho]|e\rangle = i\frac{\Omega}{2}(\langle e|\rho|g\rangle - \langle g|\rho|e\rangle) = \Omega \operatorname{Im}\langle g|\rho|e\rangle$$

Challenge: find quantum description for decay processes and combine with coherent dynamics (like Rabi oscillations).

Idea: density matrix ρ for atom+field system. Take the trace over the field and get the *reduced density matrix* ρ_A for the atom:

$$\rho_A = \operatorname{tr}_F \rho.$$

Use second order perturbation theory (“weak interaction”) combined with a separation of “slow” and “fast” timescales: “fast” = fluctuations of the field (short correlation time τ_c); “slow” = dissipative dynamics (decay) of the atom. Recall that $\tau_c \approx \hbar/k_B T \approx 10^{-12}$ s for a thermal field (“fast”) and $1/\gamma \approx 10^{-9}$ s (“slow”) for typical lifetimes. And also $1/\omega_A \approx 10^{-14}$ s for a typical optical period (“very fast”). The resulting equation is “better” than simple time-dependent perturbation theory — it allows to describe the atomic dynamics on long timescales, even longer than a decay time.

The calculation we present is due to Wigner and Weis(s?)kopf. Our presentation is inspired by the chapter IV.B of the book “Atom-photon interactions” by Cohen-Tannoudji, Dupont-Roc, and Grynberg.

Density matrix, interaction picture

Schrödinger equation for the atom+field density matrix

$$\frac{d\rho}{dt} = \frac{1}{i\hbar} [H, \rho]$$

with the Hamiltonian

$$H = H_A + H_F + H_{\text{int}}$$

Atom plus laser mode: describe both by H_A .

Interaction picture: unitary transformation

$$\rho(t) = e^{-i(H_A+H_F)t/\hbar} \tilde{\rho}(t) e^{i(H_A+H_F)t/\hbar}$$

gives Schrödinger equation

$$\frac{d\tilde{\rho}}{dt} = \frac{1}{i\hbar} [H_{\text{int}}(t), \tilde{\rho}] \quad (4.9)$$

where the time-dependence of the interaction is that of the free evolution

$$H_{\text{int}}(t) = e^{i(H_A+H_F)t/\hbar} H_{\text{int}} e^{-i(H_A+H_F)t/\hbar},$$

as we saw before. In the following, we drop the tilde to simplify the writing.

Formal solution to (4.9):

$$\rho(t + \Delta t) = \rho(t) + \frac{1}{i\hbar} \int_t^{t+\Delta t} dt_1 [H_{\text{int}}(t_1), \rho(t_1)]$$

Iteration to second order:

$$\begin{aligned} \rho(t + \Delta t) = & \rho(t) + \frac{1}{i\hbar} \int_t^{t+\Delta t} dt_1 [H_{\text{int}}(t_1), \rho(t)] \\ & - \frac{1}{\hbar^2} \int_t^{t+\Delta t} dt_1 \int_t^{t_1} dt_2 [H_{\text{int}}(t_1), [H_{\text{int}}(t_2), \rho(t_2)]] \quad (4.10) \end{aligned}$$

Reduced atom dynamics

Now take the trace over the field to find $\rho_A(t + \Delta t)$. Specify the initial conditions and the interaction.

- Initial conditions: atom+field state factorizes

$$\rho(t) = \rho_A(t) \otimes \rho_F \quad (4.11)$$

Take arbitrary state $\rho_A(t)$ for the atom and a thermal state for the field. Idea: field is a “big” system and its state changes a little when coupled to the atom.

- Interaction: as usual, electric dipole interaction

$$H_{\text{int}}(t) = -\mathbf{d}(t) \cdot \mathbf{E}(t)$$

The time-dependence is that of the free evolution.

Immediate consequence: the trace over the field of the first order term in (4.10) vanishes:

$$\text{tr}_F(H_{\text{int}}(t)\rho(t)) = -\rho_A \mathbf{d}(t) \cdot \text{tr}(\mathbf{E}(t)\rho_F) = 0$$

because the average electric field vanishes in a thermal state.

Hence, the dynamics will be given by the second order term. Note that the unknown density matrix $\rho(t_2)$ appears under the integral in (4.10). In the spirit of perturbation theory, we replace it by the initial $\rho(t)$. If we took into account corrections, they would be due to the interaction, and the double integral is already of second order in H_{int} . This approximation makes the evolution of the reduced density matrix only dependent on its value at time t , it is called the “Markov approximation” and one says that the field has an “infinitely short memory time”.

We note that the assumption (4.11), that at time t , the atom+field density matrix factorizes, cannot be true at all times. The atom-field interaction creates correlations, and these are precisely responsible for getting a nonzero result at second order. Thus, we can only approximate the full density matrix by a factorized one. The idea is that the atom-field correlations decay on the fast timescale given by the field correlation time τ_c , and that after a time step Δt , they have died out. The Markov approximation precisely assumes that this decay happens quasi-instantaneously. Note also that when we work at zero temperature, we assume that the photons the atom possibly emits “disappear” in the field reservoir (since its state remains the vacuum state). This means that the master equation cannot describe the re-absorption of spontaneously emitted radiation, as it may happen in front of a mirror or in a cavity. For this, either one must use the correct mode functions for the geometry (the van der Waals shift then appears as a second-order level shift), or one must single out the cavity mode from the field and combine it with the atom into the “system” Hamiltonian.

Field correlations

Now we have to deal with the average of two electric field operators (a first-order coherence function). For example, the first term of the double commutator involves

$$1 : \quad \text{tr}_F (E_i(t_1)E_j(t_2)\rho_F) =: C_{ij}(\tau)$$

This coherence function only depends on the time difference $\tau = t_1 - t_2$ because the field is in a stationary (in particular, thermal) state and its density matrix commutes with the field Hamiltonian. *Proof:* Under the trace, we permute cyclically and find

$$\begin{aligned} & \text{tr}_F \left(e^{iH_F t_1/\hbar} E_i e^{iH_F(t_2-t_1)/\hbar} E_j e^{-iH_F t_2/\hbar} \rho_F \right) \\ &= \text{tr}_F \left(e^{iH_F(t_1-t_2)/\hbar} E_i e^{iH_F(t_2-t_1)/\hbar} E_j \rho_F \right). \end{aligned}$$

This is a general property of the coherence function of stationary states. A field in a stationary state is thus “stationary” in the sense mentioned in the exercises (sounds consistent): its coherence function only depends on time differences. (Note, however, that we deal here with the full field operators and not a normally ordered coherence function.)

In the second term in the double commutator, the fields $\mathbf{E}(t_1)$ and $\mathbf{E}(t_2)$ appear in reverse order, leading to the coherence function $C_{ji}(t_2 - t_1) = C_{ji}(-\tau)$.

The coherence function $C_{ij}(\tau)$ of the electromagnetic field is related to the blackbody spectrum, as discussed in the exercises. In particular, in free space, it is proportional to the Kronecker δ_{ij} , and we can write $C_{ij}(\tau) = \delta_{ij}C(\tau)$. We also note that $C(-\tau) = [C(\tau)]^*$.

The four terms in the expansion of the double commutator thus give (the numbers on the left hand side simply enumerate these terms)

$$\begin{aligned} 1 : & \quad \mathbf{d}(t_1) \cdot \mathbf{d}(t_2)\rho_A C(\tau) \\ 2 : & \quad -d_i(t_1)\rho_A d_i(t_2)C(-\tau) \\ 3 : & \quad -d_i(t_2)\rho_A d_i(t_1)C(\tau) \\ 4 : & \quad \rho_A \mathbf{d}(t_2) \cdot \mathbf{d}(t_1)C(-\tau) \end{aligned}$$

Summation over the index i is understood. The terms 1, 4 are hermitean conjugates of each other. Same for the pair 2, 3.

Dipole operator

To proceed, we specify the time-dependence of the dipole operator

$$\mathbf{d}(t) = \mathbf{d}\sigma_+(t) + \mathbf{d}^*\sigma_-(t) = \mathbf{d}\sigma_+e^{i\omega_A t} + \mathbf{d}^*\sigma_-e^{-i\omega_A t}$$

where $\mathbf{d} = \langle e|\hat{\mathbf{d}}|g\rangle$ is the (complex) vector of matrix elements for the dipole operator. Noting that $\sigma_+^2 = \sigma_-^2 = 0$, we find for the product of dipole operators

$$\mathbf{d}(t_1) \cdot \mathbf{d}(t_2) = |\mathbf{d}|^2\sigma_+\sigma_-e^{i\omega_A\tau} + |\mathbf{d}|^2\sigma_-\sigma_+e^{-i\omega_A\tau}$$

The term 1 is thus equal to

$$1 : \quad C(\tau)|\mathbf{d}|^2 \left(\sigma_+\sigma_-e^{i\omega_A\tau} + \sigma_-\sigma_+e^{-i\omega_A\tau} \right) \rho_A \quad (4.12)$$

Note that it only depends on the time difference τ .

Short and long timescales

This suggests that the other time integral can be done, giving Δt . More precisely, using the new variables t_1 and τ , we can write

$$\int_t^{t+\Delta t} dt_1 \int_t^{t_1} dt_2 = \int_t^{t+\Delta t} dt_1 \int_0^{t_1-t} d\tau = \int_0^{\Delta t} d\tau \int_{t+\tau}^{t+\Delta t} dt_1$$

In the t_1, t_2 -plane, the integration domain is a triangle below the diagonal. In the last step, we have parametrized this triangle by lines of fixed τ , i.e. parallel to the diagonal, starting at $t_1 = t + \tau$ and ending at $t_1 = t + \Delta t$, as shown in figure 4.1.

We now take advantage of the short correlation time of the field: as a function of τ , the coherence function $C(\tau)$ decreases on a scale given by the coherence time τ_c . Since we are interested in the “slow” dynamics of the atom, the time interval Δt can be chosen large compared to τ_c . In this limit, we can also replace the integration domain by the parallelogram shown in fig. 4.1. This means that the integration limits for the t_1 integral change to

$$\int_0^{\Delta t} d\tau \int_{t+\tau}^{t+\Delta t} dt_1 \approx \int_0^{\Delta t} d\tau \int_t^{t+\Delta t} dt_1.$$

Since the term 1 (4.12) does not depend on t_1 , the integral over t_1 yields Δt . Term 4 (its hermitean conjugate) behaves in the same way. Such a

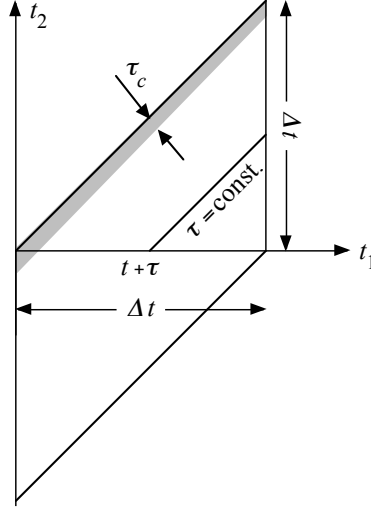


Figure 4.1: Domain of integration in the t_1, t_2 -plane. Due to the short coherence time τ_c of the field, only the gray area just below the diagonal does contribute to the integral.

simplification does not hold for the terms 2 and 3. Here, we have to invoke a different argument. Spelling out the dependence on τ and t_1 , we have

$$2: \quad -C(\tau) \left(d_i \sigma_+ e^{i\omega_A(t_1-\tau)} + d_i^* \sigma_- e^{-i\omega_A(t_1-\tau)} \right) \\ \times \rho_A \left(d_i \sigma_+ e^{i\omega_A t_1} + d_i^* \sigma_- e^{-i\omega_A t_1} \right)$$

The product gives either terms independent of t_1 or involving $e^{\pm 2i\omega_A t_1}$. If $\omega_A \Delta t \gg 1$, these oscillating terms average out when integrating over t_1 . Note that if the optical period is shorter than the correlation time τ_c of the field, this approximation is even less severe than the condition $\Delta t \gg \tau_c$ we already used. It is thus consistent to make this “secular approximation” as it is called; note its similarities to the rotating wave approximation.

Doing so, the term 2 reduces to something independent of t_1 and proportional to the squared dipole matrix element:

$$2: \quad -C(\tau) |\mathbf{d}|^2 \left(e^{i\omega_A \tau} \sigma_- \rho_A \sigma_+ + e^{-i\omega_A \tau} \sigma_+ \rho_A \sigma_- \right). \quad (4.13)$$

The t_1 integral again gives Δt .

Field spectrum

We now have to treat the integral over τ that is still left. It contains the following two integrals

$$\int_0^{\Delta t} d\tau C(\tau) e^{\pm i\omega_A \tau}.$$

(The terms with $C(-\tau)$ can be expressed in terms of the complex conjugate.) Since we work on “long timescales” $\Delta t \gg \tau_c$, the correlation function is already zero at the upper limit, and we can replace this limit by $+\infty$ without much error. We thus get a “one-sided” Fourier integral. We write the correlation function in terms of its (normal) Fourier expansion,

$$C(\tau) = \int \frac{d\omega}{2\pi} S(\omega) e^{-i\omega\tau}$$

where $S(\omega)$ is essentially the blackbody spectrum. The relation $C(-\tau) = [C(\tau)]^*$ entails that $S(\omega)$ is real. With the following formula for the τ integral

$$\int_0^{\infty} d\tau e^{i(\omega_A - \omega)\tau} = \pi\delta(\omega_A - \omega) - i\mathcal{P}\frac{1}{\omega - \omega_A}$$

(\mathcal{P} means the principal value), we find

$$\Delta t \gg \tau_c : \quad \int_0^{\Delta t} d\tau C(\tau) e^{\pm i\omega_A \tau} = \frac{1}{2}S(\pm\omega_A) - i\mathcal{P}\int \frac{d\omega}{2\pi} \frac{S(\omega)}{\omega \mp \omega_A}. \quad (4.14)$$

To simplify the notation, we shall write the second term as $-iR(\pm\omega_A)$.

The first and second terms are thus given by

$$\begin{aligned} 1 : \quad & \frac{1}{2}|\mathbf{d}|^2 (S(\omega_A)\sigma_+\sigma_- + S(-\omega_A)\sigma_-\sigma_+) \rho_A \\ & -i|\mathbf{d}|^2 (R(\omega_A)\sigma_+\sigma_- + R(-\omega_A)\sigma_-\sigma_+) \rho_A \end{aligned} \quad (4.15)$$

$$\begin{aligned} 2 : \quad & -|\mathbf{d}|^2 \left[\left(\frac{1}{2}S(\omega_A) - iR(\omega_A) \right) \sigma_-\rho_A\sigma_+ \right. \\ & \left. + \left(\frac{1}{2}S(-\omega_A) - iR(-\omega_A) \right) \sigma_+\rho_A\sigma_- \right] \end{aligned} \quad (4.16)$$

When adding term 3 (the complex conjugate of term 2), the imaginary parts involving $R(\pm\omega_A)$ drop out because the operators $\sigma_-\rho_A\sigma_+$ and $\sigma_+\rho_A\sigma_-$ are hermitean. But $R(\pm\omega_A)$ survives in terms 1 and 4.

The master equation

Coarse grained derivative We now collect our results. Recalling that the double integral reduces to something proportional to Δt , we can make a difference quotient appear on the left hand side. This gives the following “master equation”

$$\frac{\Delta\rho_A}{\Delta t} = \frac{1}{i\hbar} [H^{(2)}, \rho_A] + \mathcal{L}[\rho_A] \quad (4.17)$$

The terms on the right hand side are discussed in a minute. This difference quotient can be written as a derivative averaged over the “slow” timescale

$$\frac{\Delta\rho_A}{\Delta t} = \frac{1}{\Delta t} \int_t^{t+\Delta t} dt_1 \frac{d\rho_A(t_1)}{dt}$$

Such an average is sometimes called “coarse grained” (*grobkörnig*) because one is not interested in the rapid fluctuations that occur on timescales faster than Δt .

In the following, we focus on long timescales and use the notation $\Delta\rho/\Delta t \mapsto d\rho/dt$, not making the difference between the ordinary derivative and its coarse grained average.

Level shifts On the right hand side of the master equation (4.17), we have a “Hamiltonian-like” term that is given by

$$\begin{aligned} H^{(2)} &= -\frac{|\mathbf{d}|^2}{\hbar} (R(\omega_A)\sigma_+\sigma_- + R(-\omega_A)\sigma_-\sigma_+) \\ &= -\frac{|\mathbf{d}|^2}{\hbar} (R(\omega_A)|e\rangle\langle e| + R(-\omega_A)|g\rangle\langle g|) \end{aligned} \quad (4.18)$$

This Hamiltonian describes the energy shifts due to the coupling with the field. It is very similar to the Lamb shift we saw in chapter 3. Indeed, for the ground state, we get

$$\langle g|H^{(2)}|g\rangle = -\frac{|\mathbf{d}|^2}{\hbar} R(-\omega_A) = -\frac{|\mathbf{d}|^2}{\hbar} \mathcal{P} \int \frac{d\omega}{2\pi} \frac{S(\omega)}{\omega + \omega_A}.$$

Let us focus on the field at zero temperature, as in chapter 3. The spectrum is then given by

$$S_{ij}(\omega) = \int d\tau e^{i\omega\tau} \langle E_i(\tau) E_j(0) \rangle$$

$$\begin{aligned}
&= \sum_{\mathbf{k}\mu} E_k^2 \varepsilon_{\mathbf{k}\mu}^i \varepsilon_{\mathbf{k}\mu}^{j*} \underbrace{\langle a_{\mathbf{k}\mu} a_{\mathbf{k}\mu}^\dagger \rangle}_{=1} \int d\tau e^{i(\omega - \omega_k)\tau} \\
&= 2\pi \sum_{\mathbf{k}\mu} \delta(\omega - \omega_k) E_k^2 \varepsilon_{\mathbf{k}\mu}^i \varepsilon_{\mathbf{k}\mu}^{j*}
\end{aligned} \tag{4.19}$$

Putting this into the ground level shift, we recover the formula (??) for the Lamb shift:

$$\langle g | H^{(2)} | g \rangle = - \sum_{\mathbf{k}\mu} \frac{E_k^2 |\mathbf{d} \cdot \boldsymbol{\varepsilon}|^2}{\hbar(\omega_k + \omega_A)}. \tag{4.20}$$

As an exercise, you can work out the corresponding expression for the excited state shift. These shifts are usually ignored by incorporating them into the atomic Hamiltonian, $H_A \mapsto H_A + H^{(2)}$. In fact, the coupling to the electromagnetic field is always there and shifts the energy levels. The observed atomic transition frequency thus already contains them, and the “bare” (not shifted) frequency ω_A that we put at the beginning is not observable as such. This procedure is called “renormalization” and is the second trick to handle the infinities that occur in quantum electrodynamics.

There are quantum optics applications where the level shift is not “hidden” in a renormalized frequency and must be taken into account. For example when the atom is placed close to a mirror or other macroscopic objects. One gets a position-dependent modification of the Lamb shift, better known as van der Waals potential that enters the master equation as an additional potential: it shifts the atomic transition frequency and, more generally, the energy levels. One also finds that the atomic decay rate, discussed below, is changed in front of a mirror.

Decay rates The last term, $\mathcal{L}[\rho_A]$, of the master equation (4.17) contains the “superoperator” \mathcal{L} or “Liouvillian” that is a linear mapping of the density matrix. It cannot be written in terms of a commutator. Collecting the terms of our derivation, it has the following form

$$\begin{aligned}
\mathcal{L}[\rho_A] &= -\frac{\gamma_e}{2} \{\sigma_+ \sigma_-, \rho_A\} + \gamma_e \sigma_- \rho_A \sigma_+ \\
&\quad - \frac{\gamma_g}{2} \{\sigma_- \sigma_+, \rho_A\} + \gamma_g \sigma_+ \rho_A \sigma_-
\end{aligned} \tag{4.21}$$

where $\{A, B\} = AB + BA$ is the anti-commutator and where the rates $\gamma_{e,g}$ are given by

$$\gamma_g^e = \frac{|\mathbf{d}|^2}{\hbar^2} S(\pm\omega_A) \tag{4.22}$$

Note that the Liouvillian (4.21) also conserves the trace of the density matrix (as it must).

Rate equations

The Liouvillian superoperator \mathcal{L} describes decay processes. To see this, let us work out the equations of motion for the populations $p_{e,g}$, i.e., the diagonal elements of the density matrix. Using again $\sigma_+\sigma_- = |e\rangle\langle e|$ and $\sigma_-\sigma_+ = |g\rangle\langle g|$, we get

$$\begin{aligned}\dot{p}_e &= -\gamma_e p_e + \gamma_g p_g \\ \dot{p}_g &= -\gamma_g p_g + \gamma_e p_e\end{aligned}\tag{4.23}$$

We see here explicitly that the total population is conserved. We recover the rate equations (4.8) when $\gamma = \gamma_e$ and $\gamma_g = 0$, as is the case for a field at zero temperature. In this case, the rate γ_e is the spontaneous decay rate of the excited state. This can be seen using the field spectrum (4.19):

$$\gamma_e = \frac{2\pi}{\hbar^2} \sum_{\mathbf{k}\mu} \delta(\omega_A - \omega_k) E_k^2 |\mathbf{d} \cdot \boldsymbol{\varepsilon}|^2 = \frac{2\pi}{\hbar} \sum_{\mathbf{k}\mu} \delta(E_{e0} - E_{g1}) |\langle g; 1_{\mathbf{k}\mu} | \mathbf{d} \cdot \mathbf{E} | e; 0 \rangle|^2$$

The last expression is Fermi's Golden Rule for the transition rate between the state $|e; 0\rangle$ into the continuum of states $|g; 1_{\mathbf{k}\mu}\rangle$ due to the electric dipole coupling. At nonzero temperature, also the ground state can “decay” via the absorption of thermally excited photons.

The stationary state of the rate equations (4.23) is attained at large times and reads

$$\begin{aligned}p_e(t \rightarrow \infty) &= \frac{\gamma_g}{\gamma_g + \gamma_e} \\ p_g(t \rightarrow \infty) &= \frac{\gamma_e}{\gamma_g + \gamma_e}\end{aligned}\tag{4.24}$$

For zero temperature, we find $p_g(t \rightarrow \infty) = 1$. For finite temperature, we can expect that we recover the Boltzmann factor:

$$\frac{p_e(t \rightarrow \infty)}{p_g(t \rightarrow \infty)} = \frac{\gamma_g}{\gamma_e} = e^{-\hbar\omega_A/k_B T}$$

It is a simple exercise to show that the ground state excitation rate is proportional to the mean photon number at the transition frequency (stimulated absorption), $\gamma_g \propto \bar{n}(\omega_A)$, while for the excited state decay, stimulated emission gives $\gamma_e \propto 1 + \bar{n}(\omega_A)$. Using the mean photon number

$\bar{n}(\omega_A) = (e^{\hbar\omega_A/k_B T} - 1)^{-1}$, we indeed get the Boltzmann factor:

$$\frac{\bar{n}(\omega_A)}{1 + \bar{n}(\omega_A)} = \frac{1}{e^{\hbar\omega_A/k_B T} - 1} \frac{e^{\hbar\omega_A/k_B T} - 1}{e^{\hbar\omega_A/k_B T} - 1 + 1} = e^{-\hbar\omega_A/k_B T}.$$

Decoherence

Finally, the real benefit of our master equation is the equation of motion for the off-diagonal elements (the coherences) of the density matrix. From (4.17), we get for example

$$\frac{d\rho_{eg}}{dt} = -\frac{\gamma_e + \gamma_g}{2} \rho_{eg} \quad (4.25)$$

and its complex conjugate. We observe that the coherences decay due to the coupling to the field. This process is often called “decoherence” — quantum superpositions (with nonzero off-diagonal density matrix elements) turn into classical probabilistic alternatives (described by diagonal density matrices) due to the interaction with some “environment”. The decoherence rate is sometimes written $1/T_2$ in distinction to the rate $\gamma_e = 1/T_1$ for the populations. Note that there exist couplings where one gets a decoherence rate that is larger than the value $(\gamma_e + \gamma_g)/2$ obtained here. This value is a lower limit, except special cases (a “squeezed reservoir”, for example). In the important case of zero temperature (more precisely, $\hbar\omega_A \gg k_B T$), the “optical coherence” ρ_{eg} decays with $\gamma_e/2$, hence at half the rate of the excited state population.

4.3 Vacuum energy

4.4 Casimir interactions

Bibliography

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