Photons and other quasi-particles

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1 Plasma waves and plasmons

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"Classical physics" – actually more recent formulation than quantum physics. System is collection of many particles, each one describe with classical mechanics. Relevant forces depend on particles: may be charged, hence electric and magnetic forces appear in the equations of motion. There may be collisions when two particles come close, and then fly apart.

1.1 Statistical description

. . . needed because of the large number N of particles $\mathcal{O}(10^{23})$. From one to another "test volume", the number N varies – introduce probability distribution P.

average (expected) particle number $N_{\rm exp} = Pn$ with total number of particles n (definition of probability!).

similarly, sort particles by value of parameter y: introduce "particle density"

$$\frac{dN_{\rm exp}}{dy} = n\frac{dP}{dy} \tag{1}$$

Example: spatial interval dx or volume d^3x, leads to usual particle density

$$N(y) = \lim_{dy\to 0} \frac{dN_{\rm exp}}{dy} \tag{2}$$

and the particle number in some interval Δy

$$\int_{\Delta y} dy\, N(y) \tag{3}$$

This "macroscopic number" is the thing one actually measures.

Question: how to interpret this derivative / integral? mathematically, by using a differential quotient. But physically, the intervals dy cannot be too small, because the statistical description still needs a large number of particles (in the interval). Hence the "differential dy is still macroscopic".

Basic quantity: distribution function for point particles

$$f = \frac{d^6 N}{d^3x\, d^3v} \tag{4}$$

assuming non-relativistic particles for simplicity where the velocity is the appropriate coordinate in phase space (use the momentum in the relativistic case).

Modelling of collisions

Concept of collision cross section (Wirkungsquerschnitt) σ, actually a probability (density). Number of collisions of particle 1 over time interval δt

$$N_{\text{coll},12} = \ell_1 \sigma n_2 \tag{5}$$

with spatial density n_2 and path length $\ell_1 = v_1 \delta t$ for velocity v_1 of particle 1.

Definition: mean free path λ = characteristic path length for one collision "on average" $N_{\text{coll},12} = 1$

$$\lambda = \frac{1}{\sigma n_2} \tag{6}$$

typical collision time

$$t_{\text{coll}} = \frac{\lambda}{v} = \frac{1}{\sigma n_2 v} \tag{7}$$

Estimate for liquid water: collision cross section about a few atomic radii, gives $\sigma \sim 10^{-15}\,\text{cm}^2$. Typical particle density $10^{22}\,\text{cm}^{-3}$, hence mean free path $\sim 10^{-7}\,\text{cm}$.

Distribution function in velocity space: Maxwell-Boltzmann distribution, a Gaussian in the 3D velocity. Typically given as a function of the modulus of the velocity, gives $4\pi v^2$ times a Gaussian. This gives the equilibrium distribution under collisions between neutral particle, no other perturbations.

Examples for non-Maxwell distributions: dilute molecular cloud in space, density approx $100\,\text{cm}^{-3}$, hence the mean free path is about the distance between Earth and Sun. For a dilute gas of electrons, the collision cross section is of the order of the Thomson cross section, $\sim 10^{-24}\,\text{cm}^2$, even smaller. Hence collisions are rare, and the distribution is not Maxwellian.

1.2 Debye screening

Main example for this lecture. This is a plasma of electrons and ions, with equal number = quasi-neutral plasma. Equal average charge densities

$$en_e = Zen_i \tag{8}$$

with electron and ion densities n_e and n_i and charge state Z for ions. If this were not true, then an electric field is generated that tends to move the charges towards each other, hence neutralizing the charge imbalance.

Plot potential energy of electron vs position: a "fluctuating" function. Electron would like to be at some minimum, but its kinetic energy is typically much larger. Expand around the minimum: get a harmonic potential where the electron oscillates. Observation in a plasma: the electrons oscillate together – this is called a "collective oscillation" that happens at the so-called (electron) plasma frequency

$$\omega_{pe} = \sqrt{\frac{e^2 n_e}{\varepsilon_0 m_e}} \tag{9}$$

typical numbers: for $n_e = 1\,\mathrm{cm}^{-3}$, one gets about $60\,\mathrm{Hz}$. For the density in a metal, $10^{23}\,\mathrm{cm}^{-3}$, the frequency increases into the $100\,\mathrm{THz}$ range, comparable to atomic binding energies. (To be expected from atomic units.)

Screening: the concept of an "impurity charge", positive or negative, put into the plasma. This additional charge will attract or repel the mobile electrons. The displacement of the electrons gives a "space charge" whose sign is opposite to the impurity.

Simple calculation: assume spherical symmetry around the impurity. Poisson equation for the electric potential

$$\varepsilon_0 \nabla^2 \phi = -\rho = -\left(Q\delta(\mathbf{r}) + \rho_e\right) \tag{10}$$

where Q is the impurity charge and

$$\rho_e = -en_e + Zen_i \tag{11}$$

is the "excess electron charge" that also depends on the impurity charge Q. We do not want to solve the equations of motion for the electrons and ions to get ρ_e and adopt a statistical description. We also assume that sufficient time has elapsed so that the electron distribution has reached thermal equilibrium. For simplicity, the ions are "heavy enough" to have a density given by the equilibrium electron density that we call n_e in the following. This gives (ions positive, electrons negative with Boltzmann factor)

$$\rho_e = en_e - en_e \exp(e\phi/kT) \tag{12}$$

Assume that we can Taylor expand the Boltzmann factor. This gives, to the lowest non-vanishing order

$$\rho_e \approx -\frac{e^2 n_e \phi}{kT} \tag{13}$$

Hence the Poisson equation becomes

$$\varepsilon_0 \nabla^2 \phi = -Q\delta(\mathbf{r}) - \frac{e^2 n_e \phi}{kT} \tag{14}$$

or a Yukawa equation

$$\varepsilon_0 \left(\nabla^2 + \lambda_D^{-2}\right) \phi = -Q\delta(\mathbf{r})$$
$$\frac{1}{\lambda_D^2} = \frac{e^2 n_e}{\varepsilon_0 kT}$$

The solution to this modified Poisson equation is an exponentially suppressed Coulomb potential (also known as Yukawa potential or Debye potential)

$$\phi(r) = \frac{Q}{4\pi\varepsilon_0 r} \mathrm{e}^{-r/\lambda_D} \tag{15}$$

Typical numbers for the Debye screening length

$$\lambda_D \approx 7\,\mathrm{cm} \sqrt{\frac{T/\mathrm{K}}{n_e/\mathrm{cm}^{-3}}} \tag{16}$$

Laboratory experiments reach $T \sim 10^4\,\mathrm{K}$ and $n_e \sim 10^{15}\,\mathrm{cm}^{-3}$, and $\lambda_D \sim 1\,\mu\mathrm{m}$. Hence over a length scale larger than one micron, one "does not see" the impurity charge. (Hence the word "screening".) For collisions, this means that two electrons have to get closer than one micron to collide.

In a cosmic plasma, the Debye length $\sim 100\,\mathrm{m}$. But can our simple derivation work in such a setting? Can the thermodynamic equilibrium description be applied? Check that the average particle number "per Debye volume" $n_e \lambda_D^3$ is large enough. This number turns out to be the ratio between the thermal collision rate between two electrons and the (collective) plasma frequency.

Coulomb scattering between two electrons, mean free path λ_{ee}, collision frequency ν_{ee},

$$\lambda_{ee} = \frac{1}{n_e \sigma_{\mathrm{T}}} \left(\frac{kT}{m_e c^2} \right)^2 \tag{17}$$

$$\nu_{ee} = \frac{v_{e,\mathrm{th}}}{\lambda_{ee}} \tag{18}$$

Using the formula for the Thomson cross section, one finds

$$\nu_{ee} = \frac{\omega_{pe}}{4\pi n_e \lambda_D^3} \tag{19}$$

The number in the denominator is just the number of electrons per Debye volume. It must be large to apply thermodynamic equilibrium and the statistical description:

$$g = 4\pi n_e \lambda_D^3 \gg 1 \tag{20}$$

Under these conditions, we also have $\omega_{pe} \gg \nu_{ee}$ and hence the collective dynamics is the dominant one. This may lead to strongly non-thermal distribution functions.

In the opposite limit $g \ll 1$, two-particle collisions are dominant and lead to thermal equilibrium. (In some space applications, one gets the so-called κ-distribution.)

Next step: equation of motion for distribution function, simplified to hydrodynamics. If electric and magnetic fields are relevant, this gives magnetohydrodynamics. We are going to see collective waves for the electron density, how the propagation of light is modified (the phase velocity becomes larger than c) etc.

Question. Non-equilibrium velocity distribution may be strongly non-thermal and contain very energetic particles (energy above $10^{20}\,\mathrm{eV}$, macroscopic). How are these generated? Maybe in relativistic jets around supermassive black holes where few particles get more and more energy.

Problems. Solve Poisson-Debye equation
Thomson cross section?

1.3 Evolution of distribution function

Equations of motion for many-particle system: relevant forces and 'collective interactions'. For particles with charge q and with gravitational force

$$\dot{\mathbf{p}} = q[\mathbf{E} + \mathbf{v} \times \mathbf{B}] + \mathbf{F}_g \tag{21}$$

Relevant for dark matter which is made from unknown particles with little electromagnetic interaction (neutral, $q = 0$), but massive. Collective interactions ('collisions') happen via the gravitational field.

Distribution function

$$f = \frac{\mathrm{d}^6 N}{\mathrm{d}^3 x \mathrm{d}^3 p} \tag{22}$$

(also possible with velocities $\mathbf{v}$ instead of momenta $\mathbf{p}$) and its equation of motion. Total time derivative

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \dot{\mathbf{x}} \cdot \nabla_x f + \dot{\mathbf{p}} \cdot \nabla_p f \tag{23}$$

what other processes make f change? For example: particles appear or disappear (relevant for unstable elementary particles or when atoms change their ionization state). In other words: 'gain vs loss' formulation of collisions.

Gain: particle appears at collision point $\mathbf{x}$ with velocity $\mathbf{v}_2$ 'after collision'.

Loss: particle disappears at collision point $\mathbf{x}$ with velocity $\mathbf{v}_1$ 'before collision'.

In some cases, there are no collisions (or can be neglected). In this case, we are dealing with the

$$\textit{Vlasov equation} \qquad \frac{\mathrm{d}f}{\mathrm{d}t} = 0 \tag{24}$$

If collisions are modelled by certain gain and loss terms ('collision integrals'), we are working with the *Boltzmann equation*.

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \dot{f}_c \tag{25}$$

where $\dot{f}_c$ is a shorthand for the collision integral.

Finally: need to 'close' the description by equations that determine the force fields (electromagnetic, gravity) in terms of the particle distribution f. In electrodynamics, we are using the Maxwell equations, of course. For example, the charge and current densities are the particle-related quantities that appear there as a 'source terms'. For species of particles α with charge q_α (and a distribution function f_α for each species)

$$\rho_e = \sum_\alpha q_\alpha \int \mathrm{d}^3 p \, f_\alpha \tag{26}$$

$$\mathbf{j}_e = \sum_\alpha q_\alpha \int \mathrm{d}^3 p \, \mathbf{v} f_\alpha \tag{27}$$

For the mass density (source of gravitational fields), we have similarly

$$\varrho = \sum_\alpha m_\alpha \int \mathrm{d}^3 p \, f_\alpha \tag{28}$$

where species α has particle mass m_i.

Towards hydrodynamics

The above system of equations is complicated because of the large number of coordinates. We are going to 'forget details' and focus on quantities that arise by integrating over the momentum. (By analogy to statistics, these are called 'moments' of the distribution function.)

$$\text{moments} \qquad \int \mathrm{d}^3p \, \left\{ m \,|\, m\mathbf{v} \,|\, mv^2 \right\} f \tag{29}$$

In collisions, certain quantities are conserved, for example the (total) mass, momentum or energy. This leads to moments that vanish:

$$\int \mathrm{d}^3p \, \left\{ m \,|\, m\mathbf{v} \,|\, mv^2 \right\} \dot{f}_c \tag{30}$$

Let us focus on the mass density (first term in brackets). We multiply the Boltzmann Eq.(25) with m and integrate. Remembering that in the kinetic description, time t and momentum $\mathbf{p}$ are independent coordinates[1]we can take the time derivative outside the p-integral and find from the first term

$$\int \mathrm{d}^3p \, m \frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \int \mathrm{d}^3p \, m f = \frac{\partial \varrho}{\partial t} \tag{31}$$

the time derivative of the mass density. The second term with the particle velocity becomes $\mathbf{v} \cdot \nabla_x$ in the kinetic theory. Its moment can be re-written as

$$\int \mathrm{d}^3p \, m\mathbf{v} \cdot \nabla_x f = \nabla_x \cdot \int \mathrm{d}^3p \, m\mathbf{v} f = \nabla_x \cdot (\varrho \mathbf{V}) \tag{32}$$

where $\mathbf{V}$ is the velocity field defined by the above integral over $m\mathbf{v}f$. It can be understood as the 'average velocity' at position $\mathbf{x}$. We call it the flow velocity (*Strömungsgeschwindigkeit*). We have used that position and momenta are independent variables to take the derivative out of the integral.

Finally, the term involving the force $\mathbf{F}$ becomes after a partial integration in momentum space (we assume that the terms at infinity vanish, which is ensured when sufficiently high moments in momentum exist)

$$\int \mathrm{d}^3p \, \mathbf{F} \cdot \nabla_p f = -\int \mathrm{d}^3p \, m f \nabla_p \cdot \mathbf{F} = 0 \tag{33}$$

If we assume that the momentum dependence of the force appears only via the magnetic field, we are dealing with the divergence of $\mathbf{p} \times \mathbf{B}$ which is zero. Hence this term vanishes.

[1]In Eq.(23), the time derivatives like $\dot{\mathbf{x}}$ are a 'trajectory' or 'streamline' representation of the change in the distribution function, as if we followed a particle through phase space. But actually, we only need to know that the velocity is $\dot{\mathbf{x}} = \mathbf{p}/m$ and the force $\dot{\mathbf{p}}$ a given function of the fields or the coordinates $\mathbf{x}$.

We have found in this way the equation of continuity (conservation of mass)

$$\partial_t \varrho + \nabla_x \cdot (\varrho \mathbf{V}) = 0 \tag{34}$$

if we assume that in collisions, the total mass is conserved.

A similar calculation yields for the conservation of momentum (written in vector components, Einstein summation convention, spatial gradient $\partial_i = \partial / \partial x_i$)

$$\varrho \left(\partial_t V_j + V_i \partial_i V_j \right) + \partial_i \pi_{ij} = \mathcal{F}_j \tag{35}$$

where $\mathcal{F}_j$ is the force density on the ensemble of particles at position $\mathbf{x}$:

$$\vec{\mathcal{F}} = \int \mathrm{d}^3 p \, \mathbf{F} f \tag{36}$$

and the pressure tensor π_{ij} is the variance of the distribution function in the velocities

$$\pi_{ij} = \int \mathrm{d}^3 p \, m v_i v_j f - \varrho V_i V_j \tag{37}$$

We have used in Eq.(35) that the total momentum is conserved in collisions.

The pressure tensor is a second-order moment of the distribution function. One faces here the problem that the different moments are coupled and in principle, we have an infinite number of moments. In practice, one solves this with an approximation (a 'truncation of the hierarchy of moments'). For example, we can assume that we know how the pressure is to be calculated. This relation $P = g(\varrho)$ is called *equation of state*. A simple example is the ideal gas $P \sim k_B T \varrho / m$ where T is the gas temperature.

The simplest assumption is that the distribution is isotropic so that the integral is proportional to the unit tensor

$$\pi_{ij} = \int \mathrm{d}^3 p \, m v_i v_j f - \varrho V_i V_j = P \delta_{ij} \tag{38}$$

This leads to a pressure force proportional to the gradient ∇P.

1.4 Plasma oscillations

Assumption: there is a single wavelength and frequency, the wave extends all over space (homogeneous or bulk medium). And its amplitude is small enough so that we can linearize the Boltzmann or hydrodynamic equations. And we focus on a single spatial dimension and a longitudinal wave. We further simplify to a typical plasma, where the ions have a larger mass compared to the electrons. We assume that the ions do not move in the wave field that is oscillating. This should be OK if the frequency is high enough.

The non-perturbed situation shall be 'constant density', 'no collective motion' and 'zero pressure'. We use the index 1 for wave perturbations around this setting. For the electric field, for example,

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 = \mathbf{E}_1 = E_1 \hat{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t) \tag{39}$$

The field is assumed to point along the direction $\hat{\mathbf{k}}$ of the wave vector: this is called a 'longitudinal field'. Taking the divergence and using the Faraday equation,

$$\mu_0 \partial_t \mathbf{H}_1 = -\nabla \times \mathbf{E}_1 = \mathbf{0} \tag{40}$$

hence the wave has no magnetic component. (One also speaks of an 'electrostatic wave', although the frequency is of course nonzero.)

Coulomb equation and perturbed charge density of the electrons (the ions do not move!)

$$\varepsilon_0 \nabla \cdot \mathbf{E}_1 = \rho_e = -e n_{e,1} \tag{41}$$

Finally, the currrent density is carried by the collective motion of the electrons and since the magnetic field is zero

$$\mathbf{j}_e = \nabla \times \mathbf{H}_1 - \varepsilon_0 \partial_t \mathbf{E}_1 \tag{42}$$

we get for the wave with frequency ω

$$-e n_e \mathbf{V}_e = \mathrm{i} \varepsilon_0 \omega \mathbf{E}_1 \tag{43}$$

where we have linearized the current density and n_e is the non-perturbed electron density (fixed by the ion density).

The electric field provides the force on the electrons

$$m_e n_e \left(\partial_t \mathbf{V}_e + (\mathbf{V}_e \cdot \nabla) \mathbf{V}_e \right) = -e n_e \mathbf{E} \tag{44}$$

Since the electron velocity $\mathbf{V}_e$ is small, we neglect the convective derivative because it is of second order. Similarly, the electric force density only contains the wave field $\mathbf{E}_1$. This yields

$$-\mathrm{i} m_e n_e \omega \mathbf{V}_e = -e n_e \mathbf{E}_1 \tag{45}$$

The equation of continuity becomes similarly

$$0 = \partial_t (n_e + n_{e,1}) + \nabla \cdot (n_e \mathbf{V}_e) = -\mathrm{i}\omega n_{e,1} + \mathrm{i}\mathbf{k} \cdot n_e \mathbf{V}_e \tag{46}$$

We thus have three equations for the amplitudes $n_{e,1}$, $\mathbf{V}_e$, and $\mathbf{E}_1$. The equations are a homogeneous linear system, and we need a special relation to find a nontrivial solution. This relation is called *dispersion relation* for which we find

$$\omega^2 = \frac{e^2 n_e}{\varepsilon_0 m_e} = \omega_{p,e}^2 \,, \tag{47}$$

the (electron) plasma frequency. Observe that there is no k-dependence here. It appears in the link between the perturbed charge density and the velocity

$$n_{e,1} = n_e \frac{\mathbf{k} \cdot \mathbf{V}_e}{\omega_{p,e}} \tag{48}$$

To discuss the physics of this wave, we introduce the phase velocity

$$v_\phi = \frac{\omega}{k} = \frac{\omega_{p,e}}{k} \tag{49}$$

that slows down as the wavelength increases. But the group velocity is zero

$$v_g = \frac{\partial \omega}{\partial k} = 0 \tag{50}$$

Summary

Boltzmann equation gives the time evolution of the distribution function, at a very fundamental level.

A simplified description discards the detailed information about the momentum distribution and focuses on macroscopic fields like the mass density, velocity flow etc. These fields are defined by moments of the distribution function where the momentum coordinate is 'integrated out'. These fields describe the macroscopic hydrodynamics of the many-body system. One has to truncate the hierarchy of moments and 'close' the set of equations, using for example the equation of state that fixes the current in terms of the density. There are other approaches based on 'adiabatic conditions' where the entropy of the momentum distribution is conserved.

We have finally analyzed waves in an electric plasma, solving the coupled Boltzmann and Maxwell equations. A standard technique is based on linearization where the wave amplitude is small. The analysis yields a relation between wavelength and frequency, the dispersion relation. For a simplified setting (longitudinal wave with weak amplitude)

1.5 Transverse oscillations and plasmon polaritons

From problem session #1:

- Doppler shift for a fluid in macroscopic motion, $\omega = \omega_p + \mathbf{k} \cdot \mathbf{V}$
- If thermal motion of the plasma particles is included (see problem session), then the dispersion relation becomes
$$\omega^2 = \omega_p^2 + k^2 v_{\rm rms}^2 \tag{51}$$
where $v_{\rm rms}$ is the root mean square of the velocity distribution function. This can be related to temperature when the distribution function is thermal, but other cases are possible.

This lecture: transverse electromagnetic waves and their dispersion relation in a plasma. Need from Maxwell the curl equations. Assume for the linearization that there are no 'macroscopic' electric and magnetic fields, hence only $\mathbf{E}_1$ and $\mathbf{B}_1$ appear. (Reason for electric fields: they would be compensated by mobile charges.) (In SI units, using $1/c^2 = \varepsilon_0\mu_0$.)

$$\nabla \times \mathbf{E}_1 = -\partial_t \mathbf{B}_1 \tag{52}$$
$$\nabla \times \mathbf{B}_1 = \frac{1}{c^2}\partial_t \mathbf{E}_1 + \mu_0 \mathbf{j}_1 \tag{53}$$

The current density participates in the wave motion and modifies the dispersion relation. Build a wave equation by eliminating the magnetic field and get

$$\nabla \times (\nabla \times \mathbf{E}_1) = -\frac{1}{c^2}\partial_t^2 \mathbf{E}_1 - \mu_0 \partial_t \mathbf{j}_1 \tag{54}$$

Assume a wave field with wave vector $\mathbf{k}$ and frequency ω: replace derivatives. In the following, $\mathbf{E}_1$ etc are the (complex) amplitude vectors of the wave field.

$$[k^2\mathbf{E}_1 - \mathbf{k}(\mathbf{k} \cdot \mathbf{E}_1)] = \frac{\omega^2}{c^2}\mathbf{E}_1 + \mathrm{i}\omega\mu_0 \mathbf{j}_1 \tag{55}$$

If, as in the last lecture, we consider longitudinal waves where $\mathbf{k}$ and $\mathbf{E}_1$ are parallel, then the lhs vanishes. We focus for simplicity on the opposite case ('transverse waves') where $\mathbf{k} \perp \mathbf{E}_1$. Then on the lhs the second term vanishes, and we would find in vacuum: $k^2 = (\omega/c)^2$, the usual dispersion relation when there is no current.

In a plasma, the (transverse) electromagnetic waves have a different dispersion because of the 'participation' of the current density $\mathbf{j}_1$. It gets contributions from ions and from electrons. As before, we assume that the frequency is high enough that the ions cannot follow the wave motion. The electrons, without the wave, travel randomly in all directions, and the (average) current is zero. To get a nonzero current $\mathbf{j}_1$, we have to perturb the system. Since the average velocity (zero'th order) is zero, the linearization leads to

$$\mathbf{j}_1 = -en_0\mathbf{V}_1 \tag{56}$$

where n_0 is the equilibrium electron density (equal to the ion density by overall neutrality). Now we need a relation between the wave fields and the electron velocity field. This relation can be found by linearizing the equations of motion of the plasma, since we work in the small amplitudes. (This is a typical procedure: the 'response' of the material system is found by perturbation theory assuming weak e.m. fields.)

The equation of motion for the electron velocity is

$$m\partial_t\mathbf{V}_1 = -e\mathbf{E}_1 \tag{57}$$

we neglect the magnetic force $\mathbf{V} \times \mathbf{B}$ because it is of second order in the wave amplitude (and we neglect large-scale / macroscopic magnetic fields!).

Solving Eq.(57) in the time-harmonic regime, we get from Eq.(56) a current density

$$\mathbf{j}_1 = \mathrm{i}\frac{e^2 n_0}{m\omega}\mathbf{E}_1 \tag{58}$$

and inserting into the wave equation, we find the dispersion relation

$$k^2 = \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2} \tag{59}$$

where the plasma frequency arises as before from

$$\frac{\omega_p^2}{c^2} = \frac{\mu_0 e^2 n_0}{m} = \frac{e^2 n_0}{m\varepsilon_0 c^2} \tag{60}$$

We have thus found the following dispersion relation for transverse electromagnetic waves in a plasma

$$\omega^2 = \omega_p^2 + c^2k^2 \tag{61}$$

This leads to a phase velocity $v_\phi = \sqrt{c^2 + \omega_p^2/k^2} > c$. It is *not* a problem that this is faster than the vacuum speed of light c: it only gives the velocity for a global pattern, not of an actual signal. The group velocity is indeed smaller than c

$$v_g = \frac{c^2}{v_\phi} < c \tag{62}$$

The phase velocity plays a role, however, for certain optical phenomena: for example Huyghens' elementary spherical waves move at v_ϕ. Consider a plane wave that crosses a small cloud of electrons – after the cloud, the phase fronts are deformed in such a way that the cloud works like a defocusing lens. This phenomenon plays a role for light coming from distant sources in the Universe, for example the light pulses from a pulsar. Different frequencies are delayed with a different phase, and this has to be corrected to reconstruct the pulsar signal. For cosmic plasma densities, this happens in the radio frequency band.

Another example: what happens when a wave with $\omega < \omega_p$ tries to cross a plasma? Formally, one gets $k^2 < 0$ which leads to imaginary values of the k-vector. In this case, the wave does not enter the plasma. (In condensed matter, in a metal, say, one finds a penetration depth (sometimes called Meissner length) c/ω_p.) This plays a role for radiofrequency observations of the Sun: the corona has a plasma frequency in the MHz range. Therefore signals in this band do not originate from the Sun because the waves cannot cross the corona plasma. The signal originates from 'further out', hence the Sun appears to have a larger diameter in the radio band.

Remark on large-scale magnetic fields. Every motion of charges is deviated into a Larmor circle by the force $\mathbf{v}_1 \times \mathbf{B}$. This leads to a second characteristic frequency, the Larmor frequency $\omega_B = eB/m$. Electrons and ions move in circles with opposite orientation. This leads to different phase velocities for e.m. waves with circular polarization (rotating electric field) – electrons moving with the polarization have a larger impact on the phase of the wave. If one starts with a linearly polarized wave, one can write it as a combination of clockwise and counter-clockwise circular polarizations. The two acquire different phases, and the net result is a rotation of the linear polarization, also known as Faraday rotation.

1.6 Damping and generation of waves: Landau damping

For this last example, we come back to the collisionless (Vlasov) description of the plasma wave, but keep a kinetic theory framework where the basic object is the distribution function. For simplicity, we focus on a one-dimensional wave along the x-direction. The Boltzmann-Vlasov equation becomes (no collisions!)

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\mathbf{E}\cdot\nabla_{\mathbf{v}} f = 0 \tag{63}$$

Again we linearize $E = E_1$ and $f = f_0 + f_1$ and drop second-order terms. The zero'th order gives

$$\frac{\partial f_0}{\partial t} + v\frac{\partial f_0}{\partial x} = 0 \tag{64}$$

and the first order

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} - \frac{e}{m}E_1\frac{\partial f_0}{\partial v} = 0 \tag{65}$$

For the solution to the zero'th order, we take the equilibrium Boltzmann distribution, that we assume independent of time and space. The velocity dependence $f_0(v)$ is relevant, however, because it appears in the first-order equation. Assuming a plane wave with parameters k, ω, we replace the derivatives and get

$$-\mathrm{i}(\omega - kv)f_1 = \frac{e}{m}E_1\frac{\partial f_0}{\partial v} \tag{66}$$

The Poisson-Coulomb law relates the electric field and the charge density

$$\varepsilon_0\frac{\partial E_1}{\partial x} = -en_1 = -e\int \mathrm{d}v\, f_1 \tag{67}$$

Inserting Eq.(66), we find the relation

$$1 = \frac{e^2}{mk\varepsilon_0}\int \mathrm{d}v\frac{\partial f_0/\partial v}{kv - \omega} \tag{68}$$

where the electric field amplitude E_1 has been cancelled from both sides. On the rhs, the prefactor can be re-written in terms of the plasma frequency: $e^2/m\varepsilon_0 = \omega_p^2/n_e$. We have to handle the singularity under the integral at $kv - \omega = 0$ and come back to a complex-valued frequency $\omega_r + \mathrm{i}\omega_i$. Hence we consider waves that grow or decay in time, depending on the sign of the imaginary part ω_i. Landau has approximately evaluated the integral and finds a complex frequency for the wave

$$\omega \approx \omega_p + \frac{\mathrm{i}\pi}{2}\,\mathrm{sgn}(k)\frac{\omega_p^3}{n_e k^2}\,\frac{\partial f_0}{\partial v}\bigg|_{v=\omega_p/k} \tag{69}$$

This can be interpreted as a growth or damping mechanism: particles that move at the wave's phase velocity, $v = \omega_p/k$, contribute to the wave amplitude. If the distribution function has a negative slope $\partial f_0/\partial v < 0$, the wave is damped. In the opposite case, the wave is growing – this may happen when an additional electron beam is injected into the plasma. The wave takes energy out of the beam and eventually changes the distribution function so that a flat plateau (in velocity) appears.

1.7 Remarks on quantum aspects

For some applications, we have to describe the radiation field in terms of photons, more precisely 'polaritons' since their dispersion relation is not that of vacuum. In interactions with atoms or electrons, these polaritons behave in the same way as photons in Compton scattering, for example. For energy and momentum balances, for example, one introduces the polariton as a 'quasi-particle' with a momentum $\hbar k$ and energy $\hbar\omega$.

It may also happen that energy and momentum conservation laws can only be satisfied when two or three particles participate in a collision process. For example, transverse and longitudinal photons can be transformed one into the other.

Ideas for electrons in a metallic conductor Check estimates for Debye screening, for collision frequencies (collective or single-particle dominated?) – main difference to (astrophysical) plasmas is the degeneracy?

Check surface plasmon dispersion relation, split into complex components

2 Plasmons and surface plasmons

Carsten Henkel – 05 May / 12 May 15

We move in this section towards collective excitations of electrons in a metal, i.e., a conductor with a large electron density. The simplest model is based on the assumption that the electrons move freely. The large density forces us to consider a degenerate electron gas where one of the characteristic energy scales is given by the Fermi energy E_F instead of the thermal energy $k_B T$.

2.1 Evidence for plasmons

Experimental evidence for "plasmons" (as quasi-particles): electron energy loss spectroscopy (EELS) – electron beam with energy E_i crosses thin metal film (foil), and one observes a distribution in the final energy E_f or equivalently, in the energy loss, $p(\Delta E)$ with $\Delta E = E_i - E_f$. Typical spectra are shown in Fig. 5 from Rocca (Surf Sci Rep 1995).

One sees two peaks which can be identified with two characteristic frequencies for collective oscillations – bulk plasmon $\Delta E \approx \hbar\omega_p$ – surface plasmon $\Delta E \approx \hbar\omega_s \approx \hbar\omega_p/\sqrt{2}$ The simple scale factor $1/\sqrt{2}$ applies when the film is surrounded by vacuum. The intuitive

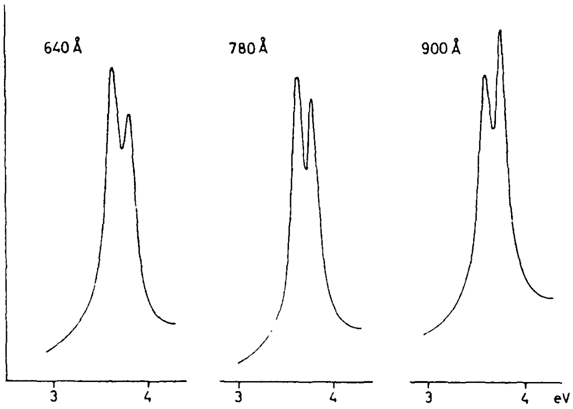

Fig. 5. EEL spectrum recorded with swift electrons (E_i = 50 keV) for thin Ag foils of different thickness at room temperature. Bulk and surface plasmons are clearly resolved, at $\hbar\omega_p$ = 3.78 eV and $\hbar\omega_{sp}$ = 3.63 eV as $\epsilon_2(\omega)$ nearly vanishes in this frequency range (from Ref. [65], used with permission).

Figure 1: Electron energy loss spectra for different thicknesses of a metal film.

interpretation is: a scattering process has happened where an electron has given an energy ΔE to a the collective plasma oscillation, this exchange is quantized in energy packets $\hbar\omega_p$. One also says that a quasi-particle "plasmon" has been created by the electron. The "surface plasmon" is studied below where we also look at collective oscillations at the interface between a metal and a medium (like vacuum).

Experimentally, one can also choose an arrangement where the dispersion relation $\omega_s(q)$ of the surface plasmon is probed. The argument q is a wave vector parallel to the surface, and it appears when momentum conservation is written down:

$$mv_i \sin\theta_i = mv_f \sin\theta_f + \hbar q \tag{70}$$

where $v_{i,f}$ are the electron velocities (related to the energies $E_{i,f}$ and the angles $\theta_{i,f}$ give the orientation of the incoming and scattered electron beam relative to the surface normal. The factor $\sin\theta$ gives the projection of the electron momentum parallel to the surface.

The interpretation of this relation is also intuitive: the incoming electron has given part of its momentum, $\hbar q$ to the surface plasmon, and escapes the surface with a different momentum (angle θ_f). Hence $\hbar q$ is the momentum carried by the quasi-particle "surface plasmon".

When the angular distribution of the scattered electron beam is scanned, one typically observes a peak for some specific angle θ_f. The corresponding momentum $\hbar q$ is "assigned" (attributed) to the surface plasmon.

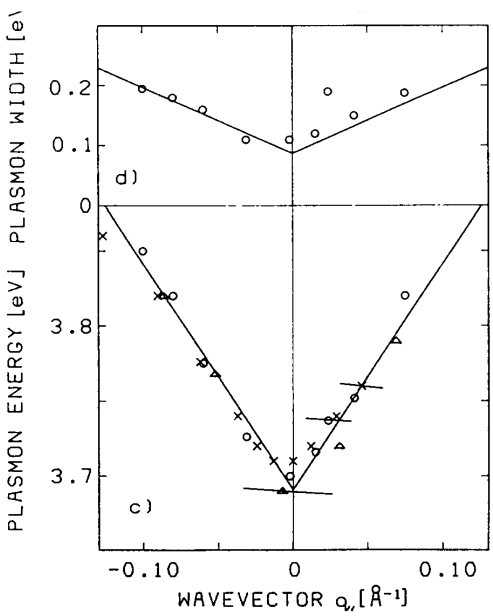

Figure 2: Dispersion relation determined by angle-resolved electron energy loss spectroscopy. This combines data from different scattering conditions: (o) $E_i = 16\,\mathrm{eV}$, $\theta_s = 81.6°$ (Δ) $E_i = 16\,\mathrm{eV}$, $\theta_s = 60.0°$ ($\times$) $E_i = 10.5\,\mathrm{eV}$, $\theta_s = 86.2°$. Fig.15 taken from Rocca (1995)..

Note that this interpretation relies on a "classical" description of the electron in terms of energy and momentum, as for a classical point particle. The factor $\hbar$ then comes in via the prescriptions of Einstein & Planck: "energy is exchanged in terms of packets $\hbar\omega$" and of de Broglie: "a wave with wave vector k carries a momentum $\hbar k$". In this way, the "particle properties" of the electron are carried over to the quasi-particle (surface) plasmon.

One could also take the reverse viewpoint and make a "full wave" description for both the electron and the plasmon. The plasmon would appear, as we found before, as a collective oscillation in hydrodynamic fields. The electron would be described as a probability wave, using quantum mechanics. The equations for momentum conservation would then be interpreted as the "phase matching" condition for the interactions between waves (language familiar in nonlinear optics).

One sees in Fig.2 a roughly linear dispersion in the wave vector q. The explanation for this will not be given in this lecture, it has to do with the details of the electronic density distribution near the surface. Let us only note that significant changes in the frequency (energy) only appear for wave vectors $\mathcal{O}(1/\text{Å})$, i.e., the collective oscillation has a wavelength of a few Ångström, comparable to the crystalline structure of the metal. In the following, we focus on much larger ("macroscopic") length scales where the dispersion relation is also significantly different.

2.2 Macroscopic theory of (surface) plasmons

In fact, we shall only deal with the frequencies ω_p, ω_s of (surface) plasmons, using "macroscopic" electrodynamics. This theoretical description combines the hydrodynamic model for the electrons in the model with the Maxwell equations. Two basic ingredients of this description are the electrical conductivity of the metal and its dielectric function.

The conductivity is the inverse of the resistance, more precisely of the specific resistance (also known as resistivity). It is thus a "local" version of Ohm's law: an electric field $\mathbf{E}$ induces a current density $\mathbf{j}$ linear in the field[2]

$$\mathbf{j} = \sigma \mathbf{E} \tag{71}$$

In general, σ will depend on frequency and on the k-vector. We shall follow a simplified treatment and neglect the k-dependence. In position space, this implies that the current density responds locally to the electric field. If σ were independent of frequency, then the current reponse would also be *instantaneous* in the field. This never happens in practice (unless one "works on slow enough time scales"), and we keep in mind that $\sigma(\omega)$ is frequency-dependent and Ohm's law Eq.(71) is only valid in frequency space. (Back in the time domain, we find that $\mathbf{j}(t)$ depends on the field $\mathbf{E}(t')$ at previous times.)

... introduce dielectric function: In the Ampère-Maxwell equation, ...

... recover (longitudinal) plasma oscillations at ω_p from $\varepsilon(\omega_p) = 0$

... macroscopic description of surface plasmons: field, charge, current distribution.

Dispersion relation from continuity of tangential fields H_y and E_x.

... discussion of dispersion relation

... see problem sheet #2 (08 May 2015)

2.3 Experiments with surface plasmons

... surface physics (electron scattering)

... biophysics: detect small variations of index of refraction (virus, cell etc.)

[2]Note the three different meanings for the letter σ: a cross section, a surface charge density, a conductivity.

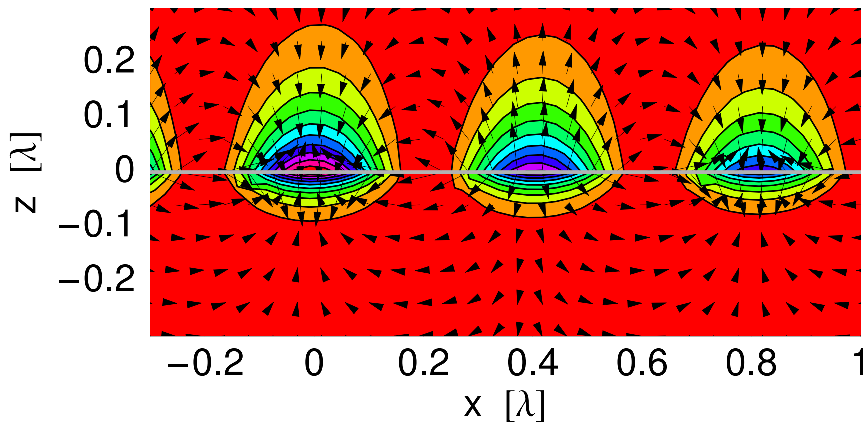

Figure 3: Field distribution of a surface plasmon at a fixed time (no averaging). Color code: intensity $\mathbf{H}^2$ of magnetic field. Arrows: electric field. Horizontal gray line: interface. Note the jump in the electric field normal to the surface, indicating the presence of a surface charge density. The surface plasmon propagates to the right, as indicated by the loss in amplitude (complex q-vector, fixed real frequency).

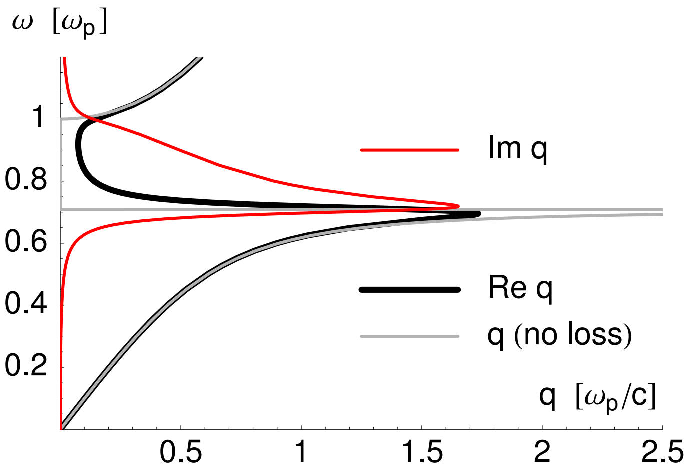

Figure 4: Dispersion relation for surface plasmons in the optical domain (q-vectors comparable to ω_p/c). For $\omega \geq \omega_p$, one recovers the dispersion relation for transverse waves in the bulk, $\omega^2 = \omega_p^2 + c^2 q^2$.

... nano photonics and plasmonics: build small optical circuits, combined with nano-electronic circuits, on length scales $100\,\mathrm{nm}$ to $10\,\mu\mathrm{m}$, limited by losses in the metal. A metallic stripe plays the role of a plasmonic waveguide.

... experiments in Potsdam: groups of S. Santer and of D. Neher, see Fig.5 taken from Papke et al. (2014).

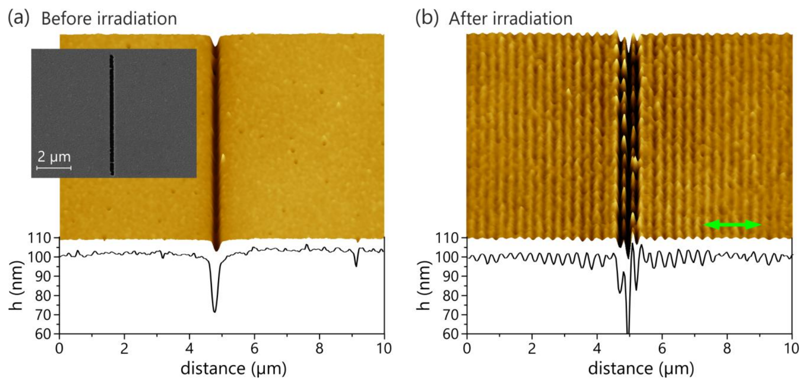

Figure 2. (a) AFM micrograph of a single nano groove etched in the silver layer and covered with a 40 nm thick trimer layer, before irradiation. Inset: SEM micrograph of the bare silver topography with etched groove. (b) AFM micrograph at the same position after irradiation from the glass side with a laser beam (wavelength $\lambda_0 = 488$ nm) intensity of 5 mW/cm^2 for 60 min. Green arrow indicates the polarization of laser beam with respect the sample topography during irradiation.

Figure 5: Topographic images of a metallic film covered with a photo-active material, before and after illumination with laser light. From the group of S. Santer, published in Papke & al (2014).

relevant physics:

thin metallic film, plasmons on top and bottom interfaces ('hybridize'). This changes the dispersion relation.

surface plasmon created by scattering from defect (groove, scratch, grating). Excitation of plasmon depends on laser polarization: need an electric field perpendicular to the surface (wall) of the defect. Experimentally easy to check by rotating the laser polarization.

periodic patterns in the topography (in the light field) arise from the interference between the scattered surface plasmon and the laser field ('hologram'). This explains the period of the observed oscillations.

References

Papke, T., Yadavalli, N. S., Henkel, C., Santer, S., 2014. Mapping a plasmonic hologram with photosensitive polymer films: standing versus propagating waves. ACS Appl. Mater. Interfaces 6 (16), 14 174–80.

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