Einführung in die Quantenoptik II

Sommersemester 2015
Carsten Henkel

Übungsaufgaben Blatt 3 Ausgabe: 07. Mai 2015
Abgabe: 20. Mai 2015

Problem 3.1 – Bloch equations (7 points)

The equations of motion for the observables σ, $\pi_e, \ldots$ of a two-level atom are called the (optical) Bloch equations. You will encounter similar equations in (electronic, nuclear ...) spin resonance.

(1) Justify in your words the approximate equation for the slowly varying amplitude $\tilde{\sigma}$ of the dipole operator (see lecture, time argument of $\tilde{\sigma}$ suppressed)

$$\frac{d\langle\tilde{\sigma}\rangle}{dt} = -\mathrm{i}(\omega_A - \omega_L)\langle\tilde{\sigma}\rangle + \mathrm{i}w\langle\tilde{a}\rangle\langle\pi_e - \pi_g\rangle - \Gamma\langle\tilde{\sigma}\rangle \tag{3.1}$$

(2) Using the interaction Hamiltonian in the resonance approximation, $V = -\hbar w(a^\dagger\sigma + \sigma^\dagger a)$, motivate that the populations evolve according to (check signs)

$$\begin{aligned} \frac{d\langle\pi_e\rangle}{dt} &= \mathrm{i}w\langle\tilde{\sigma}^\dagger\tilde{a} - \tilde{a}^\dagger\tilde{\sigma}\rangle + r_e - \gamma\langle\pi_e\rangle \\ \frac{d\langle\pi_g\rangle}{dt} &= \mathrm{i}w\langle\tilde{a}^\dagger\tilde{\sigma} - \tilde{\sigma}^\dagger\tilde{a}\rangle + \gamma\langle\pi_e\rangle - \gamma_g\langle\pi_g\rangle \end{aligned} \tag{3.2}$$

where the additional terms describe pumping and depletion (*Entleerung*).

(3) Solve the Bloch equations in the stationary limit and show that the average dipole is given by (no guarantee for factors of 2, signs etc.)

$$\langle\mathbf{d}(t)\rangle = \mathbf{d}_{ge} w \frac{r_e(1/\gamma - 1/\gamma_g)(\Delta_L - \mathrm{i}\Gamma)}{\Delta_L^2 + \Gamma^2 + 2(\Gamma/\gamma)w^2|\langle\tilde{a}\rangle|^2}\langle\tilde{a}\rangle\mathrm{e}^{-\mathrm{i}\omega_L t} + \mathrm{c.c.} \tag{3.3}$$

Discuss the frequency dependence of the corresponding complex susceptibility (keyword ‘power broadening’). Try to find a frequency and a field amplitude $\langle\tilde{a}\rangle$ such that the dipole amplitude is maximal.

Problem 3.2 – Semiclassical photon gain (7 points)

A simplified model for the (average) photon number N in a laser is given by the equation of motion

$$\frac{dN}{dt} = \frac{G_0 N}{1 + \beta N} - \kappa N \tag{3.4}$$

where κ is a loss rate (photons leave the laser cavity). The rate G_0 is called the 'small-signal amplification' (why?) and β the 'saturation parameter' (why?).

(1) This equation is nonlinear and therefore difficult to solve. Make first the weak-saturation approximation, $\beta N \ll 1$ and solve the linearized equation. Explain why the laser threshold is given by the condition 'losses = gain', $\kappa = G_0$.

(2) Give arguments that qualitatively the behaviour of the non-linear equation *below threshold* is similar to the linear case: the photon number will decay to zero, whatever its initial value.

(3) By evaluating the saturated gain term for larger N, you can qualitatively sketch the solution for the photon dynamics *above threshold*. Argue that a stationary state is reached where $N(t) \to N_\infty = (G_0 - \kappa)/\beta$ and that this state is reached 'from below' $N(t) < N_\infty$ for all t, provided $N(0) < N_\infty$.

(4) Perform a stability analysis of the stationary state (above threshold) and determine the time scale on which $N = N_\infty$ is reached. (Guess: $1/\kappa$.)

Problem 3.3 – Density operators and master equations (6 points)

A quantum system like a two-level atom where processes like spontaneous emission, dephasing, and pumping occur, cannot be modelled by a wave function (state vector) and a time-dependent Schrödinger equation. The reason is that the system is not closed. In quantum optics, a more general time evolution is provided by 'master equations' whose general structure is fixed by requirements like the preservation of positivity and linearity.

(1) The basic element is the density operator $\rho(t)$ whose time evolution corresponds to the Schrödinger picture. The expectation value of an observable A it given by the 'trace rule'

$$\langle A \rangle = \mathrm{tr}(A\rho) \tag{3.5}$$

where tr is the trace of the operator. For a two-level system, ρ can be represented by a hermitian 2×2-matrix with unit trace. Show that the population $\langle \pi_e \rangle$ and the dipole $\langle \sigma \rangle$ are given by the matrix elements ($\rho_{ab} = \langle a|\rho|b\rangle$)

$$\langle \pi_e \rangle = \rho_{ee}\,, \qquad \langle \sigma \rangle = \rho_{eg} \tag{3.6}$$

(2) For a two-level system with spontaneous decay (rate γ) and dephasing (rate κ), the master equation has the form (for details, see later in the lecture)

$$\begin{aligned} \frac{d\rho}{dt} &= -\frac{\mathrm{i}}{\hbar}[H,\rho] + \gamma(\sigma\rho\sigma^\dagger - \tfrac{1}{2}\sigma^\dagger\sigma\rho - \tfrac{1}{2}\rho\sigma^\dagger\sigma) \\ &\quad + \kappa((\pi_e - \pi_g)\rho(\pi_e - \pi_g) - \rho) \end{aligned} \tag{3.7}$$

where H is the Hamiltonian (free evolution plus coupling to light field). Check that this leads to an equation of motion for the average dipole similar to Eq.(3.1), with a dipole decay rate equal to $\Gamma = \gamma/2 + \kappa$ (no guarantee for factors 2).

(3) Show that the master equation (3.7) preserves the trace of ρ, i.e., $d(\mathrm{tr}\rho)/dt = 0$. To describe a pump or loss process, try and add terms like

$$\left.\frac{d\rho}{dt}\right|_{\mathrm{pump}} = r_e \pi_e \,, \qquad \left.\frac{d\rho}{dt}\right|_{\mathrm{depl}} = -\frac{\gamma_g}{2}(\pi_g \rho + \rho \pi_g) \tag{3.8}$$

to the master equation. Conclude that in the stationary state, we have $\langle \pi_g \rangle = r_e/\gamma_g$.