**Einführung in die Quantenoptik II**

Sommersemester 2015
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**Problem 4.1** – Phase space description of a laser (10 points)

Read the section 1.7 of the lecture notes (phase diffusion in a laser) and (i) prove Eq.(1.38) and the following equation for the correlation function

$$\langle \mathrm{e}^{-\mathrm{i}[\theta(t+\tau)-\theta(t)]} \rangle = \mathrm{e}^{\mathrm{i}\omega_L \tau} \mathrm{e}^{-D\tau} \tag{4.1}$$

where $\omega_L$ is the laser frequency and $D$ the phase diffusion coefficient.

(ii) Look up a quantum optics book and check that the rules of Eq.(1.42) for the action of $a$ and $a^\dagger$ on coherent states are correct. For an elementary consistency check, evaluate the action of the commutator $[a, a^\dagger]$ on the coherent state.

**Problem 4.2** – Glauber's photo-detector and spectrum (10 points)

Recall the equation of motion for the dipole operator of a two-level system in the resonance approximation

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\Gamma\sigma + \mathrm{i}\Delta\sigma - \mathrm{i}\mathcal{E}(t)(\pi_e - \pi_g) + \text{quantum noise} \tag{4.2}$$

where $\mathcal{E}$ is the slowly varying envelope of a scaled electric field operator

$$\mathbf{d}_{ge} \cdot \mathbf{E}(t) = \hbar\mathcal{E}(t)\mathrm{e}^{-\mathrm{i}\omega_L t} + \text{h.c.} \tag{4.3}$$

Eq.(4.2) has been 'up-graded' to an operator equation by adding 'quantum noise' whose average is zero. We shall use this equation as a simple model for a photo-detector, following the original idea of Roy Glauber (Nobel prize 2005).

(i) We shall use perturbation theory and assume that the atomic operators evolve slowly enough so that we can take $\pi_e \approx 0$ and $\pi_g \approx 1$. Solve Eq.(4.2) to get

$$\sigma(t) = \sigma(0)\,\mathrm{e}^{\mathrm{i}\Delta t - \Gamma t} + \mathrm{i}\int_0^t \mathrm{d}t'\,\mathcal{E}(t')\mathrm{e}^{\mathrm{i}\Delta(t-t') - \Gamma(t-t')} + \text{noise term} \tag{4.4}$$

(ii) Use this formula to evaluate the expectation value of the excited state projector $\sigma^\dagger(t)\sigma(t)$. You may assume that $\langle\sigma^\dagger(0)\sigma(0)\rangle = \pi_e(0) \approx 0$. Argue why the photodetector signal is proportional to the probability of finding the atom in the excited state and that this probability is proportional to the *normally ordered* field correlation function

$$G(t,t') := \langle\mathcal{E}^\dagger(t)\mathcal{E}(t')\rangle \tag{4.5}$$

(iii) An axiomatic approach to the spectrum of a quantum field is based on 'filter theory'. For a time-dependent operator (in the Heisenberg picture) $\mathcal{E}(t)$, define the quantity

$$A_\varphi := \int \mathrm{d}t\, \varphi(t)\mathcal{E}(t) \tag{4.6}$$

where $\varphi(t)$ is a 'test (or filter) function' that represents the kind of measurement that is done on the field. The spectrum $S_\mathcal{E}(\omega)$ relates the average intensity of the measured operator to the Fourier transform of the test function:

$$\langle A_\varphi^\dagger A_\varphi\rangle = \int \mathrm{d}\omega\, |\tilde{\varphi}(\omega)|^2 S_\mathcal{E}(\omega) \tag{4.7}$$

where $\tilde{\varphi}(\omega)$ is the Fourier transform of the filter function. Identify from the simple model above the test function $\varphi(t)$ and its Fourier transform. Argue that in the formal limit $\Gamma \to 0$, the expectation value $\langle\sigma^\dagger(t)\sigma(t)$ is proportional to the Fourier transform

$$\int \mathrm{d}\tau\, \mathrm{e}^{+\mathrm{i}\Delta\tau}\langle\mathcal{E}^\dagger(t+\tau)\mathcal{E}(t)\rangle =: S_\mathcal{E}(\omega) \tag{4.8}$$

which is the Wiener-Khintchine formula for the spectrum.