Einführung in die Quantenoptik II

Sommersemester 2015
Carsten Henkel

Übungsaufgaben Blatt 5

Ausgabe: 10. Juni 2015
Abgabe: 17. Juni/01. Juli 2015

Problem 5.1 – Spectra – Fourier vs Laplace (10 points)

Correlation functions $\langle B(t)A(t')\rangle$ are nearly always calculated under stationary conditions, i.e., $\langle B(t+T)A(t'+T)\rangle = \langle B(t)A(t')\rangle$ for any time shift T.
(1) Show that stationary correlations only depend on the time difference $\tau = t - t'$.
(2) Show that if A and B are hermitean conjugates of each other, $A^\dagger = B$, then

$$\int\limits_{-\infty}^{+\infty} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega\tau} \langle A^\dagger(t+\tau)A(t)\rangle = 2\,\mathrm{Re} \int\limits_{0}^{+\infty} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega\tau} \langle A^\dagger(t+\tau)A(t)\rangle \tag{5.1}$$

so that one only needs 'time-ordered (stationary) correlations'.
(3) Assume that the correlation function becomes a sum of damped exponentials (complex λ_k with $\mathrm{Re}\,\lambda_k \ge 0$ are possible)

$$\langle A^\dagger(t+\tau)A(t)\rangle = \sum_k \mathrm{e}^{-\lambda_k\tau} S_k \tag{5.2}$$

and show that the half-sided Fourier integral in Eq.(5.1) yields a spectrum which is a sum of Lorentzians. What happens for complex λ_k or when $\mathrm{Re}\,\lambda_k \to 0$?

Problem 5.2 – Bloch matrix eigenvalues (20 points)

The following steps will lead you to a discussion of the three peaks in the Mollow triplet.

(1) Start with the driven atom Hamiltonian in the rotating- wave approximation

$$H_{\mathrm{AL}} = \frac{\hbar\omega_A}{2}\sigma_3 - \frac{\hbar}{2}\left(\Omega^* \mathrm{e}^{\mathrm{i}\omega_L t}\sigma + \sigma^\dagger\Omega\mathrm{e}^{-\mathrm{i}\omega_L t}\right) \tag{5.3}$$

and consider the following representation of the density operator ($\rho_{ab} = \langle a|\rho|b\rangle$)

$$\rho_{eg} = \frac{u + \mathrm{i}v}{2}\mathrm{e}^{-\mathrm{i}\omega_L t}, \qquad \rho_{ee} - \rho_{gg} = w\,, \qquad \rho_{ee} + \rho_{gg} = s \tag{5.4}$$

Taking the usual damping scheme for coherences and populations, show that these real coefficients evolve according to

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0 \tag{5.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\Gamma & -\Delta & 0 \\ \Delta & -\Gamma & -\Omega \\ 0 & \Omega & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} - \gamma \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} \tag{5.6}$$

with the detuning $\Delta = \omega_L - \omega_A$.

(2) Call 'Bloch matrix' B the 3×3-matrix appearing here. Show that its eigenvalues $-\lambda$ are solutions of the cubic polynomial

$$Q(\lambda) = (\lambda - \gamma)(\lambda - \Gamma)^2 + (\lambda - \gamma)\Delta^2 + (\lambda - \Gamma)\Omega^2 \tag{5.7}$$

(3) *Kurvendiskussion*: show that one eigenvalue, λ_1, is always real with $\lambda_1 \geq 0$. Show that for small enough Δ, Ω, the other two $\lambda_{2,3}$ are real as well, but become complex conjugates in the opposite case. Find the critical point where $\lambda_2 = \lambda_3$ is a double eigenvalue. Show that in all cases, $\mathrm{Re}\,\lambda_k \geq 0$. You may use that $\Gamma \geq \gamma/2$.

Problem 5.3 – Quantum jump correlations (10 points)

Consider a two-level system in the 'incoherent limit' where only populations are nonzero in the density matrix with the master equation

$$\frac{d\rho_{ee}}{dt} = -\gamma\rho_{ee} + R\rho_{gg} \tag{5.8}$$

(1) Give an interpretation of the parameters γ and R. (2) Write down the equation of motion for ρ_{gg}. (3) Calculate the inversion $\langle\sigma_z\rangle$ in the stationary state and make a sketch of the time-dependent function $\langle\sigma_z(t)\rangle$ for your favorite initial condition. (4) Derive, in the spirit of the regression formula, a differential equation for the correlation function

$$C_z(t) = \lim_{t' \to \infty} \langle \sigma_z(t' + t)\sigma_z(t')\rangle \tag{5.9}$$

(5) Make a sketch of $C_z(t)$ and of the corresponding spectrum. (6) Give a physical interpretation of $C_z(t)$ using the language of 'quantum jumps' between the states $|g\rangle$ and $|e\rangle$.

The solution to Eq.(5.8) is a sum of a constant and a damped exponential.