Theoretical Physics
- Introduction to General Relativity (Summer 2016) -

Problem set No 1

Emission 04.04.16 – Digestion 08.04.16

▷ Problem 1 (Ballistic trajectories and universal curvature)

If you throw a ball or shoot a bullet, the ball and the bullet are said to move on a ballisitic trajectory. In the XZ-plane (with X the horizontal, and Z the vertical)

$$x(t) = x_0 + u_0 t\,, \qquad z(t) = z_0 + w_0 t - \frac{1}{2} g t^2 \tag{1}$$

with x_0, z_0 the coordinates of the launch position, u_0, w_0 the horizontal and vertical components of the launch velocity, and $g \approx 10\mathrm{m/s}^2$ the earth gravitational acceleration.

Consider the case in which the baseball velocity horizontal component is given by $u_0 = 5\mathrm{m/s}$, while for the bullet $u_0 = 500\mathrm{m/s}$. The vertical components w_0 are adjusted so that the ball and the bullet propagate over the same horizontal distance $l = 10\mathrm{m}$.

(a) Determine the maximal height and time of flight of the two bodies.

(b) Sketch the spatial trajectories (inverted parabola) of the two bodies, and convince yourself that the radius of curvature at the respective maxima is completely different.

(c) Sketch the world-lines of the two bodies in a space-time diagram with time coordinate ct. Confirm that the radius of curvature of the world-lines for sufficiently small launch-velocities is given by

$$R = \frac{c^2}{g} \tag{2}$$

which for earth gravitational acceleration amounts to $R \approx 9 \times 10^{15}\mathrm{m}$, that is approximately one light-year, $1\mathrm{Lyr} = 9,46 \times 10^{15}\mathrm{m}$.

Remark: Evidently, the curvature of the worldlines is (1) independent of the mass of the bodies, and (2) independent of their initial velocity. The curvature is a universal – that is geometric – characteristic of the earth gravitational field. From the point of view of general relativity, the worldlines are the geodesics of a curved space-time. They are the "straightest possible paths", and the fact that they appear curved is merely a result of the chosen coordinates. Remember, that the earth surface constitutes an "accelerated reference frame", and thus the XYt-coordinates are coordinates of an accelerated frame. For more see the problem set on Rindler space-time . . .

▷ Problem 2 (Dark Star, Black Hole)

In the wake of Newtons theory of gravitation, it was pointed out by John Michell (1724–1793) and Pierre Simon de Laplace (1749–1827), that the light emitted by a star can't escape the star's gravitational field if only the star radius is smaller than its so called *gravitational radius*,

$$R_{\mathrm{G}} := \frac{2GM}{c^2}\,, \tag{3}$$

where M is the mass of the star, R its radius, G the gravitational constant, and c the speed of the emitted light. For an observer at large enough a distance, a star with radius $R < R_{\mathrm{G}}$ appears invisible or black, respectively.

(a) What would be the Laplace argument?

(b) How does the gravitational radius of the earth compare with the gravitational radius of the sun or a neutron star?

In general relativity, the gravitational radius is promoted the Schwarzschild radius. The Schwarzschild radius defines the event horizon of a spherical symmetric mass distribution. If all the mass is concentrated on a spatial region with radius smaller than the Schwarzschild radius, it is called a black hole. Nothing can escape a black hole, and all what travels beyond the event horizon is lost forever (except for mass resp. energy, electric charge and angular momentum, the totals of which define the properties of the black hole.)

(c) It looks like black hole is nothing but the Laplace dark star. In screening the popular accounts of black holes – what could be possible differences?

(d) Under the assumption of a spherically symmetric universe which came from a "big bang" what is its average mass-energy density given the universe age of approx. 14×10^9years?

Remark: What?? How could we possibly estimate the mass-energy density from the age of the universe? Well – because the universe could be a "dark star" (if seen – better: not seen – from the outside), and if its surface expands with the speed of light (what else?) ...

▷ **Problem 3 (Planck Scale)**

In gravity theory, a spherical mass distribution defines a length scale GM/c^2 (half the gravitational radius, see Eq. (3). But also the quantum mechanics of a point mass comes with a characteristic length scale – the Compton wavelength

$$\lambda\!\!\!^{-}_{\mathrm{C}} = \hbar/(Mc) \tag{6}$$

with $\hbar$ the Planck-constant, and M the mass of the particle.

Confronted with two basic frameworks, gravitational physics and quantum mechanics, each endowed with its own length-scale, there is a length scale, called the Planck length, where both the framworks meet

$$\ell_{\mathrm{Pl}} = \sqrt{G\hbar/c^3} \approx 1.62 \times 10^{-35}\mathrm{m}\,. \tag{7}$$

(a) Derive Eq. (7).

Evidently, the Planck length is independent of the mass. It is a universal constant like G, $\hbar$ or c. It is conjectured, that the Planck length defines the lower boundary for “classicality” or “smoothness” of space time: on length scales below the Planck length, space times is assumed to be “foamy” or in some other way “quantic”. The precise menaing of such metaphors is currently explored in the Max-Planck Institute for gravitational physics and other places.

Upon division by c, the Planck length defines a time scale

$$t_{\mathrm{Pl}} := \sqrt{G\hbar/c^5} = 5.39 \times 10^{-44}\mathrm{s}\,, \tag{8}$$

via $E = \hbar\omega \propto \hbar t_{\mathrm{Pl}}^{-1}$ an energy scale

$$E_{\mathrm{Pl}} = \sqrt{\hbar c^5/G} = 1.22 \times 10^{19}\mathrm{GeV}\,, \tag{9}$$

and via $E = mc^2$ a mass scale

$$m_{\mathrm{Pl}} = \sqrt{\hbar c/G} = 2.18 \times 10^{-8}\mathrm{kg}\,. \tag{10}$$

(b) Show that the mass of a black hole is at least of the order of the Planck mass.
Hint: Try to localise a black hole, but remember the Heisenberg uncertainty relation . . .

(c) Just before the Large Hadron Collider (LHC) was fired up some years ago, there was a heated debate on the potential breeding of black holes which would swallow up the earth if not the entire cosmos. The debate was brought before the European Court of Law, and it was decided that the LHC could proceed as planned. Try to reconstruct the plaintiffs’ argument, and design a defense strategy for the defendant.

▷ Problem 4 (Strong Equivalence Principle)

According to the strong equivalence principle, all rest masses and interaction energies of the constituents of a celestial body contribute to its gravitational and inertial mass in an equal manner.

Estimate the contributions of the gravitational energies and elektromagnetic binding energies to the mass of the earth.