Einführung in die Quantenoptik II

Sommersemester 2016
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Übungsaufgaben Blatt 2 Ausgabe: 04. Mai 2016
Abgabe: 11. Mai 2016

Problem 2.1 – Working with phase-space distributions (15 points)

We have introduced characteristic functions χ_s as an alternative way to describe a quantum state (density operator ρ):

$$\chi_s(z, z^*) = \mathrm{Tr}[\rho D(z)\, \mathrm{e}^{s|z|^2/2}] \tag{2.1}$$

where $D(z) = \exp(za^\dagger - z^* a)$ is Glauber's displacement operator and the parameter s follows the correspondence

$$\begin{aligned} s = 1 &\quad \text{P-function} \\ s = 0 &\quad \text{Wigner function} \\ s = -1 &\quad \text{Q-function} \end{aligned} \tag{2.2}$$

(1) Show that for $s = 1$ ($s = -1$), one gets normally (anti-normally) ordered expectations

$$\chi_1(z, z^*) = \langle : D(z) : \rangle = \langle \mathrm{e}^{za^\dagger}\, \mathrm{e}^{-z^* a} \rangle \tag{2.3}$$

$$\chi_{-1}(z, z^*) = \langle \ddagger D(z) \ddagger \rangle\, \mathrm{e}^{-|z|^2} = \langle \mathrm{e}^{-z^* a}\, \mathrm{e}^{za^\dagger} \rangle\, \mathrm{e}^{-|z|^2} \tag{2.4}$$

The expansion of these functions around $z = 0$ gives the averages of ordered moments of the $a^\dagger$ and a operators.

(2) When switching from the characteristic function to the phase-space distribution, the following Fourier integrals are useful:

$$\int \frac{\mathrm{d}^2 z}{\pi^2}\, \mathrm{e}^{-z\alpha^* + z^*\alpha} = \delta^{(2)}(\alpha) \tag{2.5}$$

$$\int \frac{\mathrm{d}^2 z}{\pi\, b}\, \mathrm{e}^{-|z|^2/b}\, \mathrm{e}^{-z\alpha^* + z^*\alpha} = \mathrm{e}^{-|\alpha|^2 b} \tag{2.6}$$

(The first formula is the $b \to \infty$ limit of the second.) Prove these formulas with the notation $z = (q + \mathrm{i}k)/\sqrt{2}$ and $\alpha = (x + \mathrm{i}p)/\sqrt{2}$. With this notation, $\mathrm{d}^2 z = \mathrm{d}q\, \mathrm{d}k/2$ and $\delta^{(2)}(\alpha) = 2\delta(x)\delta(p)$ (why?).

(3) We have seen the Q-function of a thermal state,

$$Q_T(\alpha) = \frac{\mathrm{e}^{-|\alpha|^2/(1+\bar{n})}}{\pi(1+\bar{n})}\,. \tag{2.7}$$

We shall see in the lecture that its characteristic function is given by the Fourier integral

$$\chi_{-1,T}(z) = \int \mathrm{d}^2\alpha\, \mathrm{e}^{z\alpha^* - z^*\alpha} Q_T(\alpha) \tag{2.8}$$

Compute this, multiply with a suitable factor and get the 'characteristic Wigner function':

$$\chi_{0,T}(z) = \mathrm{e}^{-|z|^2(\frac{1}{2}+\bar{n})} \tag{2.9}$$

The back-transformation or inversion of Eq.(2.8) yields

$$W_T(\alpha) = \frac{\mathrm{e}^{-|\alpha|^2/(\frac{1}{2}+\bar{n})}}{\pi(\frac{1}{2}+\bar{n})} \tag{2.10}$$

Or in words: the Wigner function is the same gaussian as the Q-function, but from its variance a term $\frac{1}{2}$ has been subtracted. For the P-function, remove another $\frac{1}{2}$ from the variance.

(4) Observe that the back-transform for the Wigner and Q-functions is 'better behaved' compared to that for the P-function. As an example, you can start with the P-function's characteristic χ_{-1} for a 'simple state' like the vacuum state. The 'most pathological' P-functions arise with number states, while the Fourier integral still exists for the other two functions (why?).

Problem 2.2 – Wigner function of thermal state (5 points)

In many textbooks, you can find the following formula for the Wigner function in terms of a pure state in the coordinate representation (wave function $\psi(x)$, we have re-introduced $\hbar = 1$):

$$W(x,p) = \int \frac{\mathrm{d}s}{2\pi\hbar}\, \psi^*(x - s/2)\psi(x + s/2)\, \mathrm{e}^{-\mathrm{i}ps/\hbar} \tag{2.11}$$

Fix the relation between α and the phase-space coordinates x, p by considering the vacuum state (oscillator ground state, Eq.(2.10) for $\bar{n} = 0$). For an oscillator in thermal equilibrium, the Wigner functions must be averaged with probabilities $p_n \sim \mathrm{e}^{-n\hbar\omega/k_BT}$ and Hermite polynomials in $\psi_n(x)$. The corresponding formula is quite cumbersome to evaluate. Without evaluating the entire sum over n, check that this approach is consistent with Eq.(2.10) by considering the low-temperature limit $\hbar\omega \gg k_BT$.