Einführung in die Quantenoptik II

Sommersemester 2016
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Problem 3.1 – Displacing states (5 points)

We have introduced the Wigner function as the Fourier transform of its characteristic function

$$\chi_0(z) = \langle D(z) \rangle = \langle \exp(z a^\dagger - z^* a) \rangle \tag{3.1}$$
$$W(q) = \int \frac{\mathrm{d}^2 z}{\pi^2} \exp(z^* q - z q^*) \chi_0(z)\,, \quad q = \frac{x + \mathrm{i}p}{\sqrt{2}} \tag{3.2}$$

where the average is taken in some state $|\Psi\rangle$ (or a density operator ρ). Show (CBH formula) that the action of a displacement operator $D(\alpha)$ on this state yields a phase shift of the characteristic function

$$|\Psi\rangle \mapsto D(\alpha)|\Psi\rangle\,, \qquad \chi_0(z) \mapsto \chi_0(z)\, \mathrm{e}^{z\alpha^* - z^*\alpha} \tag{3.3}$$

This implies a displacement of the Wigner function in the phase plane: $W(q) \mapsto W(q - \alpha)$.

Problem 3.2 – Squeezing states (10 points)

The squeezing transformation S transforms the mode operators a, $a^\dagger$ in the following way

$$a \mapsto S^\dagger a S = \mu a + \nu a^\dagger \tag{3.4}$$

(1) Compute $S^\dagger a^\dagger S$ and show in two ways that the commutator is preserved

$$[S^\dagger a S, S^\dagger a^\dagger S] = [\mu a + \nu a^\dagger, \ldots] = 1 \tag{3.5}$$

provided the 'hyperbolic identity' holds: $|\mu|^2 - |\nu|^2 = 1$.

(2) The squeezed vacuum state $|\mu, \nu\rangle$ is constructed in such a way that it is the vacuum state of the transformed operator, $S^\dagger a S |\mu, \nu\rangle = 0$. Using Eq.(3.4), show that $|\mu, \nu\rangle$ has a nonzero overlap c_n with even number states $|0\rangle$, $|2\rangle$, $\ldots$ For this reason, the squeezed vacuum is also a popular model for 'biphoton

states' created by down-conversion (a nonlinear process that creates a pair of 'signal' photons from one 'pump photon').

(3) Consider the quadrature operators and find complex parameters μ, ν such that X, P transform according to the linear maps

$$\begin{pmatrix} X \\ P \end{pmatrix} \mapsto M \begin{pmatrix} X \\ P \end{pmatrix}, \qquad M = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{e}^{\eta} & 0 \\ 0 & \mathrm{e}^{-\eta} \end{pmatrix} \tag{3.6}$$

In the textbooks, this calculation is often done with the parameter choice $\mu = \cosh\xi$, $\nu = \mathrm{e}^{\mathrm{i}\varphi}\sinh\xi$.

(4) There is a classical interpretation to maps like (3.6) as linear transformations in phase space. For this, consider the area (why this name?)

$$\vec{q}_1 \wedge \vec{q}_2 := p_1 x_2 - p_2 x_1 \tag{3.7}$$

spanned by two vectors $\vec{q}_1 = (x_1, p_1)^{\mathsf{T}}$ and $\vec{q}_2 = (x_2, p_2)^{\mathsf{T}}$. This area is related to the Poisson bracket, and it is called 'symplectic form' in mathematics. Check that the matrices M are such that the area is preserved

$$(M\vec{q}_1) \wedge (M\vec{q}_2) = \vec{q}_1 \wedge \vec{q}_2 \tag{3.8}$$

Such maps are called 'canonical' or 'symplectic'. They form a group, the so-called symplectic group.[1]

Observe that with the identification $\vec{Q} = (X, P)^{\mathsf{T}} \leftrightarrow a = (X + \mathrm{i}P)/\sqrt{2}$, the exponent of the displacement operator $D(z)$ and of the Fourier integral (3.1) can be written as an 'operator symplectic form'

$$za^\dagger - z^* a = \mathrm{i}\,\vec{x} \wedge \vec{Q}\,, \qquad \vec{x} = \ldots \tag{3.9}$$

$$zq^* - z^* q = \mathrm{i}\,\vec{x} \wedge \vec{q} \tag{3.10}$$

(5) Conclude that the action of a squeezing transformation S on the characteristic function and the Wigner function is simply

$$\chi_0(z) \mapsto \chi_0(M^{-1}z)\,, \qquad W(\vec{q}) \mapsto W(M^{-1}\vec{q}) \tag{3.11}$$

where the complex number $M^{-1}z$ and the phase-space vector $M^{-1}\vec{x}$ are related in the same way as z and $\vec{x}$ in item(4) above.

[1] Denoted Sp(n) or Sp$(n, \mathbb{R})$ where n is the dimension of phase space (even). For $n = 2$, the symplectic group is formed by all real 2×2 matrices M with $\det M = 1$, also known as SL$(2, \mathbb{R})$.