Einführung in die Quantenoptik II

Sommersemester 2016
Carsten Henkel

Übungsaufgaben Blatt 4 Ausgabe: 01. Mai 2016
Abgabe: 08. Juni 2016

Problem 4.1 – Phase diffusion model (8 points)

The process of 'phase diffusion' is a simple example of a Fokker-Planck equation. Consider an oscillatory signal whose phase $\phi = \phi(t)$ is drifting, following a random walk. (i) Write a sentence about the meaning of the distribution function $P(\phi, t)$. Choose and describe the initial distribution $P(\phi, 0)$ and justify that for large times,

$$\lim_{t\to\infty} P(\phi, t) = \frac{1}{2\pi} \tag{4.1}$$

(ii) The random walk dynamics of the phase can be described by the drift-diffusion equation

$$\frac{\partial P}{\partial t} - \omega_L \frac{\partial P}{\partial \phi} = D \frac{\partial^2 P}{\partial \phi^2} \tag{4.2}$$

Check that ω_L has the dimension of a frequency and D that of a diffusion coefficient. Transform into the 'co-moving frame'

$$P(\phi, t) = \tilde{P}(\varphi, t)\,, \quad \phi = \varphi - \omega_L t \tag{4.3}$$

and show that the solution can be written in the form

$$\tilde{P}(\varphi, t) = \sum_{n\in\mathbb{Z}} p_n\, \mathrm{e}^{\mathrm{i}n\varphi - n^2 D t} \tag{4.4}$$

(iii) Conclude that for a fixed-amplitude signal, the following average holds

$$\langle \mathrm{e}^{\mathrm{i}\phi(t)} \rangle = \langle \mathrm{e}^{\mathrm{i}\phi(0)} \rangle \mathrm{e}^{-\mathrm{i}\omega_L t} \mathrm{e}^{-Dt} \tag{4.5}$$

Hint. Study first from Eq.(4.4) the phase distribution at $t = 0$.

Problem 4.2 – From master to Fokker-Planck equation (12 points)

In the lecture (and in many quantum optics books), we have found the following rules for the action of bosonic operators on coherent state projectors:

$$\begin{aligned} a|\alpha\rangle\langle\alpha| &= \alpha|\alpha\rangle\langle\alpha \\ |\alpha\rangle\langle\alpha|a^\dagger &= \alpha^*|\alpha\rangle\langle\alpha| \\ a^\dagger|\alpha\rangle\langle\alpha| &= (\alpha^* + \partial_\alpha)|\alpha\rangle\langle\alpha| \\ |\alpha\rangle\langle\alpha|a &= (\alpha + \partial_{\alpha^*})|\alpha\rangle\langle\alpha| \end{aligned} \tag{4.6}$$

(i) To find an equation of motion for the P-function, an integration by parts (*partielle Integration*) is applied. Show that this gives, for example

$$
\begin{aligned}
a^\dagger a \rho &= \int \mathrm{d}^2\alpha \, |\alpha\rangle\langle\alpha| (\alpha^* - \partial_\alpha) \, [\alpha P] \\
a^\dagger \rho a &= \int \mathrm{d}^2\alpha \, |\alpha\rangle\langle\alpha| (\alpha^* - \partial_\alpha)(\alpha - \partial_{\alpha^*}) P
\end{aligned}
\tag{4.7}
$$

(ii) Check that this 'operator calculus' is consistent with the law $(a^\dagger \rho) a = a^\dagger(\rho a)$ because the two differential operators $(\alpha^* - \partial_\alpha)$ and $(\alpha - \partial_{\alpha^*})$ commute. Check also that the first line in (4.7) and its counterpart for $a a^\dagger \rho$ are consistent with the commutator $[a, a^\dagger]$.

(iii) A reasonable master equation for a laser has the following form

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\mathrm{i}\omega_{\mathrm{c}} \left[a^\dagger a, \, \rho\right] + \kappa \, a \rho a^\dagger - \frac{\kappa}{2} \left\{a^\dagger a, \, \rho\right\} \\
+ G(a^\dagger a) \, a^\dagger \rho a - \frac{G(a^\dagger a)}{2} \left\{a a^\dagger, \, \rho\right\} ,
\end{aligned}
\tag{4.8}
$$

which is approximate because the nonlinear gain operator $G(a^\dagger a)$ should be handled more carefully. Analyze for example, whether the trace of ρ is conserved by the three terms 'free evolution', 'cavity loss', and 'gain'.

(iv) Neglect first gain saturation and show that the equation resulting from (4.8) is (no guarantee for factors $1/2$, please find the first term)

$$
\begin{aligned}
\frac{\partial}{\partial t} P(\alpha, \, \alpha^*) &= -\mathrm{i}\omega_c \{\ldots\} P + \frac{1}{2} \{\partial_\alpha (\kappa - G)\alpha + \partial_{\alpha^*}(\kappa - G)\alpha^*\} P \\
&\quad + \frac{G}{4} \partial_\alpha \partial_{\alpha^*} P,
\end{aligned}
\tag{4.9}
$$

(The derivatives act on everything to their right, including P.) An approximate way to include gain saturation is to replace $G \mapsto G(|\alpha|^2) = G_0/(1 + B|\alpha|^2)$ and put it between the two derivatives in the last term. This term generates diffusion in phase-space. In the steady state, diffusion coefficient is, in order of magnitude, $D \sim G(\bar{n}) \sim \kappa$ where $\bar{n}$ is the average photon number.