Einführung in die Quantenoptik II

Sommersemester 2016
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Problem 6.1 – Light beam operators (10 points)

An efficient description of light beams in free space can be based on the following expansion of the field operator

$$\mathbf{E}(\mathbf{r}, t) = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \sum_{mn} \mathcal{P}_{mn}(x, y)\, \hat{\mathbf{e}}_{mn}\, \mathrm{e}^{\mathrm{i}(k_z z - \omega t)} a_{mn,\omega} + \text{h.c.} \tag{6.1}$$

where $\mathcal{P}_{mn}(x, y)$ describes the transverse modes of the beam, $\hat{\mathbf{e}}_{mn}$ is a unit polarization vector, and the commutation relation is

$$\left[a_{m'n',\omega'},\, a^\dagger_{mn,\omega}\right] = 2\pi\delta(\omega' - \omega)\delta_{m'm}\delta_{n'n} \tag{6.2}$$

(i) Analyze the physical dimension of $\mathcal{P}_{mn}$ and check that with the scaling (S is the typical cross-section of the beam)

$$\mathcal{P} \simeq \left(\frac{\hbar\omega}{\varepsilon_0 c S}\right)^{1/2} \times \text{(dimensionless function)} \tag{6.3}$$

the average $\langle a^\dagger_{mn,\omega} a_{mn,\omega}\rangle$ can be interpreted as a spectral photon number (number of photons per frequency band).

(ii) Consider a beam in a coherent state with a frequency spectrum sufficiently narrow so that we can forget about the frequency dependence of $\mathcal{P}_{mn}$. We also pick a single transverse mode $m, n = 0, 0$ and drop the mode indices. Define the instantaneous photon flux ($: \ldots :$ = normal order)

$$I(t) = : \left| \int \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\omega t} a_\omega \right|^2 : \tag{6.4}$$

and show that the intensity correlation function is

$$\frac{1}{2}\langle I(t)I(t') + I(t')I(t)\rangle = \langle I(t)\rangle\langle I(t')\rangle + \langle I(t)\rangle\delta(t - t') \tag{6.5}$$

where the average flux is $\langle I(t)\rangle = |\mathcal{A}(t)|^2$ with

$$\mathcal{A}(t) = \int \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\omega t} \alpha_\omega \tag{6.6}$$

The second term in Eq.(6.5) is called 'shot noise' (*Schrotrauschen*): it expresses that the light flux is not continuous, but carried by photon packets. For gravitational wave detectors like LIGO and VIRGO, this noise level limits the detector sensitivity at high frequencies (unless squeezed light beams are used).

Problem 6.2 – Anti-bunching (10 points)

In the lecture, we have found the quantum regression formula for correlation functions

$$\lim_{t\to\infty} \langle A(t')B(t)\rangle = \mathrm{tr}\Big\{AP(t-t'|B\rho_{\mathrm{ss}})\Big\} \tag{6.7}$$

where $P(\tau|\varrho)$ solves the master equation with initial condition $P(0|\varrho) = \varrho$ and ρ_{ss} is the stationary state (density operator) of the system.

(i) This formalism generalizes a master equation to 'skew' objects like $B\rho_{\mathrm{ss}}$ that are not, strictly speaking, density operators. Show that also skew objects can be evolved by checking the construction

$$\begin{aligned} B\rho_{\mathrm{ss}} &= \frac{1}{b^*}\big\{(b+B)\rho_{\mathrm{ss}}(b^*+B^\dagger) - (b-B)\rho_{\mathrm{ss}}(b^*-B^\dagger) \\ &\quad + \mathrm{i}(\mathrm{i}b+B)\rho_{\mathrm{ss}}(-\mathrm{i}b^*+B^\dagger) - \mathrm{i}(\mathrm{i}b-B)\rho_{\mathrm{ss}}(-\mathrm{i}b^*-B^\dagger)\big\} \end{aligned} \tag{6.8}$$

Is each of the terms a density operator? Does the solution of the master equation depend linearly on the initial conditions?

(ii) Consider the excited state projector $\pi_e = \sigma^\dagger\sigma$. Construct from the regression formula a way to compute the time and normal ordered correlation function ($t' \geq t$)

$$C(t'-t)\langle :\pi_e(t')\pi_e(t):\rangle = \langle\sigma^\dagger(t)\sigma^\dagger(t')\sigma(t')\sigma(t)\rangle \tag{6.9}$$

and show that this does *not* involve the skew operator construction of (i). Argue that $C(0) = 0$: this is called 'anti-bunching'.