Einführung in die Quantenoptik II

Sommersemester 2017
Carsten Henkel

Übungsaufgaben Blatt 1

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Problem 1.1 – Coherent states (14 points)

In quantum optics, coherent (or Glauber) states are very popular. They are used to represent as closely as possible a field mode in a 'classical state'. We use the notation of the lecture with $|\alpha\rangle$ being the normalized eigenstate of a: $a|\alpha\rangle = \alpha|\alpha\rangle$, $\langle\alpha|\alpha\rangle = 1$.

(o) [5 bonus points:] Show that there are no eigenstates of the creation operator $a^\dagger$.

(i) The coherent state has the property that quadrature fluctuations do not depend on the phase angle θ and are at minimum uncertainty:

$$X_\theta := \frac{a\,\mathrm{e}^{-\mathrm{i}\theta} + a^\dagger\,\mathrm{e}^{\mathrm{i}\theta}}{\sqrt{2}}, \qquad \langle\alpha|\Delta X_\theta^2|\alpha\rangle = \frac{1}{2} \tag{1.1}$$

(ii) Show that coherent states are not stationary, but evolve under the free field Hamiltonian according to

$$U(t)|\alpha\rangle = \mathrm{e}^{\mathrm{i}\phi(t)}|\alpha\,\mathrm{e}^{-\mathrm{i}\omega t}\rangle \tag{1.2}$$

where ω is the eigenfrequency of the mode. Calculate the phase $\phi(t)$ and find out under which conditions it remains unobservable. Comment on the sentence: 'Coherent states evolve according to the classical equations of motion of an oscillator' by working out the expectations $\langle\alpha(t)|X_\theta|\alpha(t)\rangle$ for $\theta = 0, \pi/2$.

(iv) Coherent states are not orthogonal (why?): show that for $\alpha, \beta \in \mathbb{C}$,

$$\text{not orthogonal:} \quad \langle\alpha|\beta\rangle = \mathrm{e}^{\mathrm{i}\phi(\alpha,\beta)}\exp(-\tfrac{1}{2}|\alpha-\beta|^2) \tag{1.3}$$

where the phase $\phi(\alpha,\beta) = \mathrm{Im}(\alpha^*\beta)$ is antisymmetric under the exchange $\alpha \leftrightarrow \beta$ ('as it must be' – why?).

(v) Coherent states are 'overcomplete': using an integral $\mathrm{d}^2\alpha = \mathrm{d}x\,\mathrm{d}y$ (with $\alpha = x + \mathrm{i}y$) in the complex plane, show that

$$\text{overcomplete:} \quad \int \mathrm{d}^2\alpha\,|\alpha\rangle\langle\alpha| = \pi\sum_{n=0}^{\infty}|n\rangle\langle n| = \pi\mathbb{1} \tag{1.4}$$

The word 'overcomplete' appears because of the factor $\pi > 1$.

Hint. Polar coordinates in the complex plane $\alpha = r\,\mathrm{e}^{\mathrm{i}\varphi}$ are very useful here.

(vi) The Glauber displacement operators $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ have been given this name because they 'displace' the mode operators by a complex number:

$$D^\dagger(\alpha) a D(\alpha) = a + \alpha \tag{1.5}$$

Prove this equation by replacing $\alpha \mapsto \alpha t$ and deriving a differential equation for both sides with respect to t.

(vii) Using the previous results and the Campbell-Baker-Hausdorff identity

$$\mathrm{e}^{A+B}\mathrm{e}^{\frac{1}{2}[A,B]} = \mathrm{e}^{A}\mathrm{e}^{B}\,, \qquad \text{provided } [A,B] \text{ commutes with } A \text{ and } B, \tag{1.6}$$

prove the following formulas for the displacement operators

$$\begin{aligned}
D(\alpha)|0\rangle &= |\alpha\rangle && (1.7)\\
D^\dagger(\alpha) &= D(-\alpha) = D^{-1}(\alpha) && (1.8)\\
D(\alpha) &= \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \exp(\alpha a^\dagger) \exp(-\alpha^* a) && (1.9)\\
D(\alpha)D(\beta) &= \mathrm{e}^{\mathrm{i}\phi(\alpha,\beta)} D(\alpha+\beta) && (1.10)\\
\exp(\mathrm{i}\theta a^\dagger a) D(\alpha) \exp(-\mathrm{i}\theta a^\dagger a) &= D(\alpha\,\mathrm{e}^{\mathrm{i}\theta}) \quad \text{or} \quad D(\alpha\,\mathrm{e}^{-\mathrm{i}\theta}) && (1.11)\\
D^\dagger(\alpha) \exp(\mathrm{i}\theta a^\dagger a) D(\alpha) &= \exp[\mathrm{i}\theta(a^\dagger + \alpha^*)(a+\alpha)] && (1.12)
\end{aligned}$$

Problem 1.2 – Quasi-probabilities (6 points)

The phase-space representation provides very useful pictures for quantum states and their time evolution. Coherent states play an important role here.

(i) P-function (Glauber-Sudarshan).

The P-function provides an expansion of a density operator in terms of projectors onto coherent states:

$$\rho = \int \mathrm{d}^2\alpha\, P(\alpha, \alpha^*) |\alpha\rangle\langle\alpha| \tag{1.13}$$

In the following, we write for brevity $P(\alpha)$.

This representation is not the most general one because it may be necessary to include 'skew projectors' of the type $|\alpha\rangle\langle\beta|$. This leads to the 'positive P-distribution'. For some tips and tricks, see *Numerical representation of quantum states in the positive-P and Wigner representations* by M. K. Olsen and A. S. Bradley, *Opt. Commun.* **282** (2009) 3924.

Show that a normally ordered operator product can be averaged 'in the intuitive way' with the P-function

$$\langle a^{\dagger m} a^n \rangle_\rho = \mathrm{tr}\big(a^{\dagger m} a^n \, \rho\big) = \int \mathrm{d}^2\alpha \, \alpha^{*m} \alpha^n P(\alpha) \tag{1.14}$$

For a coherent state, $\rho = |\beta\rangle\langle\beta|$, argue that $P(\alpha) = \delta^{(2)}(\alpha - \beta)$. (This is trivial, right?)

(ii) Q-function (Husimi).
Here, we deal with anti-normally ordered averages that are 'intuitive':

$$\langle a^m a^{\dagger n} \rangle_\rho = \int \mathrm{d}^2\alpha \, \alpha^m \alpha^{*n} Q(\alpha) \tag{1.15}$$

Show that this formula is consistent with the definition

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle \tag{1.16}$$

using the over-completeness relation (1.4). Show that $1/\pi \geq Q(\alpha) \geq 0$ and calculate the Q-function of the vacuum state:

$$Q_{\mathrm{vac}}(\alpha) = \frac{\mathrm{e}^{-|\alpha|^2}}{\pi} \tag{1.17}$$

Show that P- and Q-functions are related by

$$Q(\alpha) = \int \frac{\mathrm{d}^2\beta}{\pi} \, \mathrm{e}^{-|\alpha - \beta|^2} P(\beta) \tag{1.18}$$

and check that this is a 'Gaussian convolution'. Intuitive interpretation?