Einführung in die Quantenoptik II

Sommersemester 2017
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Problem 2.1 – Linear maps on phase space (10 points)

In classical mechanics, we can combine the canonical coordinates into a vector $\vec{q} = (x, p)^{\mathsf{T}}$ and consider the area spanned by two vectors:

$$\vec{q}_1 \wedge \vec{q}_2 := p_1 x_2 - p_2 x_1 \tag{2.1}$$

This map is called 'symplectic form' in mathematics (it is bilinear and antisymmetric), it is related to the Poisson bracket.

(a) Now consider matrices M such that the area is preserved

$$(M\vec{q}_1) \wedge (M\vec{q}_2) = \vec{q}_1 \wedge \vec{q}_2 \tag{2.2}$$

for all pairs $\{\vec{q}_1, \vec{q}_2\}$. Show that this is true for all matrices with $\det M = 1$. The condition (2.2) defines linear maps that are called 'canonical' or 'symplectic'.

(b) Choose two from the following three matrices and show that they are symplectic:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix}, \qquad \begin{pmatrix} \mathrm{e}^{\xi} & 0 \\ 0 & \mathrm{e}^{-\xi} \end{pmatrix}. \tag{2.3}$$

Sketch the images $M\vec{q}_1$, $M\vec{q}_2$ of two vectors $\vec{q}_1$, $\vec{q}_2$ in phase space.

(c) Consider the following mapping from the complex plane to phase space

$$f : z \mapsto \vec{q} = (\mathrm{Re}\, z, \mathrm{Im}\, z)\sqrt{2}\,, \qquad f^{-1} : \vec{q} \mapsto z = (x + \mathrm{i}p)/\sqrt{2} \tag{2.4}$$

and check that the symplectic form corresponds to a 'Fourier phase'

$$z a^{\dagger} - z^{*} a = \mathrm{i}\vec{q} \wedge \vec{Q} \tag{2.5}$$

(where $\vec{q} = f(z)$ and $\vec{Q} = f(a)$). We have seen this 'Fourier phase' in the displacement operator, for example: $D(z) = \exp(\mathrm{i}\vec{q} \wedge \vec{Q})$ with the operator-valued 'canonical coordinate vector' $\vec{Q}$.

(d) This language is useful to describe the action of a displacement operator on certain phase-space functions that characterise a quantum state. Consider the following construction

$$\chi(z) := \langle D(z) \rangle = \langle \exp(z a^\dagger - z^* a) \rangle \tag{2.6}$$

$$W(\vec{q}) := \int \frac{\mathrm{d}^2 z}{\pi^2} \exp(z^* q - z q^*) \chi(z) , \quad q = f^{-1}(\vec{q}) \tag{2.7}$$

where the average is taken in some state $|\Psi\rangle$ (or a density operator ρ). From the CBH formula, it is easy to show that the action of a displacement operator $D(\alpha)$ on this state yields a phase shift

$$|\Psi\rangle \mapsto D(\alpha)|\Psi\rangle \quad \Rightarrow \quad \chi(z) \mapsto \chi(z)\, \mathrm{e}^{z\alpha^* - z^* \alpha} \tag{2.8}$$

Please check that this implies a displacement of the Wigner function in the phase plane: $W(\vec{q}) \mapsto W(\vec{q} - \vec{a})$ where $\vec{a} = f(\alpha)$ is the 'phase-space image' of the complex parameter α. (No guarantee for the sign in $-\vec{a}$.)

The canonical maps (or matrices) form a group, the so-called symplectic group, denoted Sp(n) or Sp($n, \mathbb{R}$) where n is the dimension of phase space (even). It is lucky coincidence in two dimensions ($n = 2$) that Sp(2) is formed by all real 2×2 matrices M with $\det M = 1$, also known as SL($2, \mathbb{R}$).

Problem 2.2 – Something qualitative for the laser (10 points)

In this exercise, you are asked to collect (or recall) some basic information about the laser. We are going to approach 'quantum questions' at the end.

(a) A running laser is a system with a constant 'energy throughput'. For a typical laser pointer and a conventional laboratory laser, find out typical numbers: optical output power or intensity, input power, and 'efficiency'.

(b) A key concept is the 'la' in the laser: light amplification by an 'active medium'. The simplest model is based on a medium with two energy levels and populations n_g, n_e (particles per unit volume). A simple description would start from the following equation for the time evolution of the laser light intensity I

$$\frac{\mathrm{d}I}{\mathrm{d}t} = G(I) I - \kappa I \tag{2.9}$$

Justify the names 'gain' and 'loss' for $G(I)$ and κ. Motivate why the gain can be modelled by the following two expressions (keyword: saturation)

$$G(I) = A(n_e - n_g) = \frac{G_0}{1 + \beta I} \tag{2.10}$$

Using Eqs.(2.9, 2.10), find the stationary intensity and make a sketch as a function of the 'linear gain' G_0 (keyword: laser threshold).

(c) In the quantum theory of the laser à la Scully & Lamb [M. Sargent III and M. O. Scully, "Theory of Laser Operation" in *Laser Handbook* vol. 1 (North-Holland 1972), p.45–114], the probability of finding n photons in the laser cavity is given by the following formula

$$p_n = N \prod_{k=0}^{n-1} \frac{G_0/\kappa}{1 + Bk} \tag{2.11}$$

where N is a normalisation factor. Make a sketch of this distribution and show that the maximum probability occurs for $n_* \approx (G_0 - \kappa)/(\kappa B)$. Compare this result to item (b).

(d) Write a program that evaluates the mean photon number $\bar{n}$ and its variance from the distribution p_n of item (c) and plot the two as a function of the linear gain G_0. A conventional normalisation for the variance is the 'Mandel parameter' $Q = (\Delta n)^2/\bar{n}$. If you got everything right, Q shows a peak at the laser threshold – this behaviour is typical for a phase transition. [5 Bonus points. In case you want to take small values $B \approx 10^{-2}$, say, it's better to work with the logarithm of p_n.]