Einführung in die Quantenoptik II

Sommersemester 2017
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Problem 4.1 – Lorenz model (10 points)

In Chapter 18.8 of Mandel & Wolf, the Lorenz model is introduced as a simplified version of the dynamics of a laser coupled to its active medium. (i) Check from the Bloch equations discussed in the lecture that for the (complex) polarization P of the medium and its inversion (real) N, the following equations are reasonable

$$\partial_t P = (\mathrm{i}\Delta - \Gamma)P - \mathrm{i}gN\mathcal{E} \tag{4.1}$$

$$\partial_t N = -\gamma(N - N_s) + g\,\mathrm{Im}[P^*\mathcal{E}] \tag{4.2}$$

$$\partial_t \mathcal{E} = -\kappa\mathcal{E} + \mathrm{i}G_0 P \tag{4.3}$$

where in the last line, we also added the dynamics of the (complex) laser amplitude $\mathcal{E}$ (slowly varying, omitting the carrier $\mathrm{e}^{-\mathrm{i}\omega_L t}$).

(ii) Show that by using suitable units for P, N, $\mathcal{E}$ and the phase transformation $P(t) = P'(t)\,\mathrm{e}^{-\mathrm{i}\delta t}$, the equations can be brought into the following form (Python code, writing F instead of $\mathcal{E}$)

```
Pdot = -(gammaP + 1j*delta)*P - 1j*gammaI*(F*inv);
invDot = -2*gammaI*inv + pumpRate + gammaI*imag(conj(F)*P);
Fdot = -(kappa + 1j*deltaC)*F + 1j*kappa*P;
```

(iii) On the lecture web site, you can download a Python script that solves the Bloch-Lorenz equations numerically. Examples of output look like Fig.4.1. Write a few lines about the interpretation of the left and right plot.

(iv) Play with the parameters of the Python script and study the influence of the initial data on the dynamics. Try to find a formula for the pump threshold. Make a close-up scan near the laser threshold and check the change in the time scale to reach the stationary state.

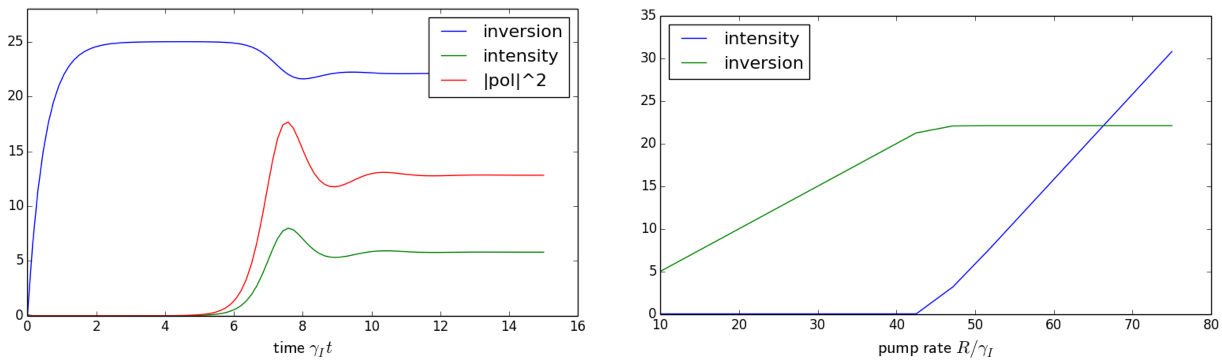

Figure 4.1: (left) Transient laser observables after onset, based on a three-component Bloch-Lorenz model. Intensity: $|\mathcal{E}|^2$, `|pol|^2`: $|P|^2$. (right) Stationary observables as a function of the pump rate.

Problem 4.2 – From master to Fokker-Planck equation (12 points)

In the lecture (and in many quantum optics books), you can find the following rules for the action of ladder operators on coherent state projectors:

$$
\begin{aligned}
a|\alpha\rangle\langle\alpha| &= \alpha|\alpha\rangle\langle\alpha \\
|\alpha\rangle\langle\alpha|a^\dagger &= \alpha^*|\alpha\rangle\langle\alpha| \\
a^\dagger|\alpha\rangle\langle\alpha| &= (\alpha^* + \partial_\alpha)|\alpha\rangle\langle\alpha| \\
|\alpha\rangle\langle\alpha|a &= (\alpha + \partial_{\alpha^*})|\alpha\rangle\langle\alpha|
\end{aligned}
\tag{4.4}
$$

where ∂_α, ∂_{α^*} are derivatives. By construction, $\partial_\alpha \alpha^* = 0$.

These rules are used to transform a master equation for the density operator into a Fokker-Planck equation for the P-function (that depends on α and α^* although this is not written explicitly).

(i) Using the representation of ρ in terms of $P(\alpha)$, apply an integration by parts (*partielle Integration*) to show that, for example

$$
\begin{aligned}
a^\dagger a\rho &= \int \mathrm{d}^2\alpha\, |\alpha\rangle\langle\alpha|(\alpha^* - \partial_\alpha)\,[\alpha P] \\
a^\dagger \rho a &= \int \mathrm{d}^2\alpha\, |\alpha\rangle\langle\alpha|(\alpha^* - \partial_\alpha)(\alpha - \partial_{\alpha^*})P
\end{aligned}
\tag{4.5}
$$

(ii) Check that this 'operator calculus' is consistent with the law $(a^\dagger\rho)a = a^\dagger(\rho a)$ because the two differential operators $(\alpha^* - \partial_\alpha)$ and $(\alpha - \partial_{\alpha^*})$ commute. Check also that the first line in (4.5) and its counterpart for $aa^\dagger\rho$ are consistent with the commutator $[a, a^\dagger]$.

(iii) A reasonable master equation for a laser has the following form

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\mathrm{i}\omega_{\mathrm{c}}\left[a^\dagger a,\, \rho\right] + \kappa\, a\rho a^\dagger - \frac{\kappa}{2}\left\{a^\dagger a,\, \rho\right\}$$
$$+G(a^\dagger a)\, a^\dagger\rho a - \frac{G(a^\dagger a)}{2}\left\{a a^\dagger,\, \rho\right\}\,, \tag{4.6}$$

which is approximate because the nonlinear gain operator $G(a^\dagger a)$ should be handled more carefully. Analyze for example, whether the trace of ρ is conserved by the three terms 'free evolution', 'cavity loss', and 'gain'.

(iv) Neglect first gain saturation and show that the equation resulting from (4.6) is (no guarantee for factors $1/2$, please find the first term)

$$\begin{aligned}\frac{\partial}{\partial t}P(\alpha,\,\alpha^*) &= -\mathrm{i}\omega_c\{\ldots\}P + \frac{1}{2}\{\partial_\alpha(\kappa - G)\alpha + \partial_{\alpha^*}(\kappa - G)\alpha^*\}P \\ &\quad + \frac{G}{4}\partial_\alpha\partial_{\alpha^*}P,\end{aligned} \tag{4.7}$$

(The derivatives act on everything to their right, including P.) An approximate way to include gain saturation is to replace $G \mapsto G(|\alpha|^2) = G_0/(1 + B|\alpha|^2)$ and put it between the two derivatives in the last term. This term generates diffusion in phase-space. In the steady state, diffusion coefficient is, in order of magnitude, $D \sim G(\bar{n}) \sim \kappa$ where $\bar{n}$ is the average photon number.