Einführung in die Quantenoptik II

Sommersemester 2017

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Übungsaufgaben Blatt 5

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Problem 5.1 – Phase diffusion model (10 points)

The process of 'phase diffusion' is a simple example of a Fokker-Planck equation. Consider an oscillatory signal whose phase $\phi = \phi(t)$ is drifting, following a random walk. (i) Write a sentence about the meaning of the distribution function $P(\phi, t)$. Choose and describe an initial distribution $P(\phi, 0)$ and justify that for large times,

$$\lim_{t\to\infty} P(\phi, t) = \frac{1}{2\pi} \tag{5.1}$$

(ii) The random walk dynamics of the phase can be described by the drift-diffusion equation

$$\frac{\partial P}{\partial t} - \omega_L \frac{\partial P}{\partial \phi} = D \frac{\partial^2 P}{\partial \phi^2} \tag{5.2}$$

Check that ω_L has the dimension of a frequency and D that of a diffusion coefficient. Transform into the 'co-moving frame' (signs to be double-checked)

$$P(\phi, t) = \tilde{P}(\varphi, t)\,, \quad \phi = \varphi - \omega_L t \tag{5.3}$$

and show that the solution can be written in the form

$$\tilde{P}(\varphi, t) = \sum_{n\in\mathbb{Z}} w_n \, \mathrm{e}^{\mathrm{i}n\varphi - n^2 D t} \tag{5.4}$$

How would you compute the expansion coefficients w_n? (iii) Conclude that for a fixed-amplitude signal, the following average holds

$$\langle \mathrm{e}^{\mathrm{i}\phi(t)} \rangle = \langle \mathrm{e}^{\mathrm{i}\phi(0)} \rangle \mathrm{e}^{-\mathrm{i}\omega_L t} \mathrm{e}^{-Dt} \tag{5.5}$$

Hint. Study first from Eq.(5.4) the phase distribution at $t = 0$.

Problem 5.2 – Correlations (10 points)

We have seen in the lecture the Wiener-Khintchine theorem,

$$S_a(\omega) = \int \mathrm{d}\tau \, \mathrm{e}^{-\mathrm{i}\omega\tau} \langle a^\dagger(t+\tau) a(t) \rangle \tag{5.6}$$

written here for a normally ordered 'quantum signal' $a(t)$. In this problem, we are going to play with these concepts using simple examples.

(i) Imagine that $a(t)$ is the amplitude of a freely evolving field mode, without damping. Write down the solution to the time-dependent operator $a(t)$ in the Heisenberg picture for a given 'initial value operator' $a = a(0)$. Show that the autocorrelation function $\langle a^\dagger(t+\tau)a(t)\rangle$ is proportional to the average photon number $\bar{n}$ (at time $t = 0$), and that the spectrum $S_a(\omega)$ is strictly monochromatic.

(ii) Master equations provide tools to compute the 'forward evolution' of a quantum system – indeed, because of losses, the 'forward' and 'backward' directions are no longer equivalent. Imagine that you know $C_a(\tau) = \langle a^\dagger(t+\tau)a(t)\rangle$ for $\tau \geq 0$ from some theory and that this correlation function is 'statistically stationary' (i.e.: it does not depend on the value of t). Show that in this fairly useful case

$$C_a(-\tau) = [C_a(\tau)]^* \tag{5.7}$$

so that

$$S_a(\omega) = 2\,\mathrm{Re}\int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega\tau} C_a(\tau) \tag{5.8}$$

(iii) A typical result of a master equation is a correlation function with an exponential decay:

$$\tau \geq 0: \quad C_a(\tau) = \bar{n}\,\mathrm{e}^{\mathrm{i}\omega_c\tau - \kappa\tau} \tag{5.9}$$

Write two sentences to explain the meaning of the parameters ω_c and κ and compute the spectrum $S_a(\omega)$. The spectrum shows a peak (a 'spectral line'): which position/frequency, which width/linewidth, which form/lineshape?