Einführung in die Quantenoptik II

Sommersemester 2017
Carsten Henkel

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Problem 7.1 – Squeezing spectra (12 points)

We have learned that squeezing has to do with reduced fluctations (variances) for some quadratures of an oscillating signal. In 1981, Walls & Zoller [*Phys. Rev. Lett.* **47** (1981) 709] argued that the dipole operator of a two-level atom, $\sigma = (\sigma_1 + \mathrm{i}\sigma_2)/2$, is squeezed if one of the two quadratures has a variance below the Heisenberg limit. In this exercise, you play with this concept and explore its generalisation to a squeezing spectrum.

(1) Introduce a general dipole quadrature

$$\sigma_\theta = \mathrm{e}^{\mathrm{i}\theta}\sigma + \mathrm{e}^{-\mathrm{i}\theta}\sigma^\dagger \tag{7.1}$$

and show that

$$\left[\sigma_\theta, \sigma_{\theta+\pi/2}\right] = 2\mathrm{i}\sigma_3 \tag{7.2}$$

According to Walls and Zoller, the quadrature σ_θ is squeezed provided that

$$(\Delta\sigma_\theta)^2 \le |\langle\sigma_3\rangle| \tag{7.3}$$

where $(\Delta\sigma_\theta)^2 = 1 - \langle\sigma_\theta\rangle^2$. Make a sketch of the region in the $\langle\sigma_\theta\rangle\langle\sigma_3\rangle$-plane (a cut of the Bloch sphere) where Eq.(7.3) is satisfied.

(2) With the Python script 'mollow-regression.html' on the lecture web site, you can solve the Bloch equations for an arbitrary initial state. The output is arranged in the order

$$\texttt{pg} = \langle\pi_g(\tau)\rangle\,, \quad \texttt{pe} = \langle\pi_e(\tau)\rangle\,, \quad \texttt{sigma} = \langle\sigma(\tau)\rangle\,, \quad \texttt{sigmaDag} = \langle\sigma^\dagger(\tau)\rangle \tag{7.4}$$

In combination with the regression formula, the script provides any correlation functions of the two-level atom. For squeezing, we are interested in

$$C_\theta(\tau) = \lim_{t\to\infty}\langle\sigma_\theta(t+\tau)\sigma_\theta(t)\rangle \tag{7.5}$$

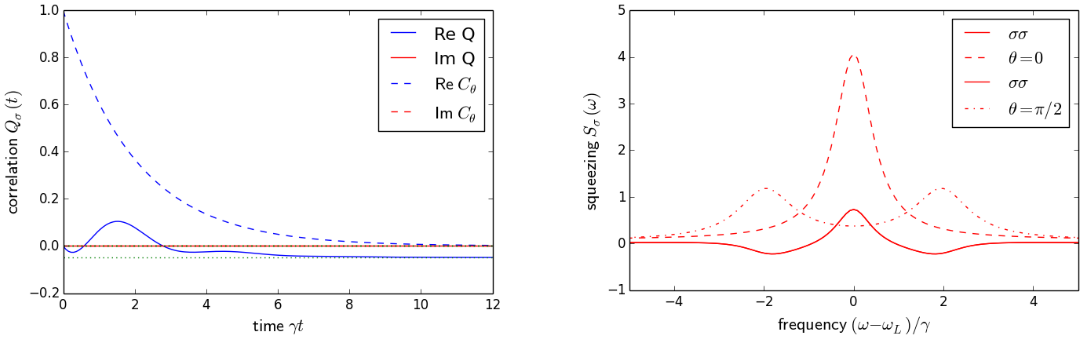
and its spectrum

$$S_\theta(\omega) = \int\limits_{-\infty}^{+\infty} \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega\tau} C_\theta(\tau) \tag{7.6}$$

Set up the formulas that you need for this spectrum in terms of initial states and expectation values (7.4) returned by the numerical Bloch solver (for $\tau \geq 0$ only). Check that the script correctly implements these formulas and correct as needed. In the python script, you can also see the correlation function

$$Q(\tau) = \lim_{t \to \infty} \langle \sigma(\tau + t) \sigma(t) \rangle \tag{7.7}$$

Explain the initial values $Q(0) = 0$ and $C_\theta(0) = 1$.



The spectra of Q (label '$\sigma\sigma$') and C_θ are shown above for quadrature angles $\theta = 0$ and $\theta = \frac{1}{2}\pi$, for a laser drive on resonance with a real and positive Rabi frequency. From the pictures, one gets: in the central band of the Mollow triplet, the light 'fluctuates more in phase with the laser', while for the red and blue sidebands, the fluctuations are mainly 'out of phase'.

(3) Play with the parameters in the script to answer a few of the following questions:

– what is the influence of the phase of the Rabi frequency on the squeezing spectra?

– how do the spectra change when the laser is off resonance? for a fixed frequency ω in the spectrum, which quadrature shows the largest squeezing?

– Walls and Zoller found that the quadrature σ_1 is squeezed when $\Delta^2 > \frac{1}{2}\Omega^2 + \frac{1}{4}\gamma^2$: can you check this numerically? (Note: no spectra required for this problem, real $\Omega > 0$.)

– Open question: how would you generalise the squeezing condition (7.3) into the spectral domain so that squeezing appears for

$$S_\theta(\omega) \leq \text{some threshold} \tag{7.8}$$

where the threshold may be frequency-dependent?

Problem 7.2 – Quantum Langevin equations (8 points)

Paul Langevin and Albert Einstein introduced the concept of a 'fluctuating force' into the modelling of Brownian motion. This has made Einstein's 1905 paper "Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen" [*Ann. Phys. (Leipzig)* **322** (1905) 549–60] the one with the most citations (3995 according to WebOfScience).

The quantum version of the Langevin equations can be used to characterise fluctuations of a quantum system around a stationary state. The simplest example is a damped cavity mode where the equations look like

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -(\mathrm{i}\omega + \frac{\kappa}{2})a + \xi(t) \tag{7.9}$$

Here, the Langevin 'force' $\xi(t)$ has the properties

$$\langle \xi(t) \rangle = 0\,, \quad \langle \xi^\dagger(t)\xi(t') \rangle = 0\,, \quad \langle \xi(t)\xi^\dagger(t') \rangle = \kappa\delta(t - t') \tag{7.10}$$

(1) Solve Eq.(7.9) using the technique of 'variation of constants'. Take the average to find the damped oscillator solution

$$\langle a(t) \rangle = \langle a(t) \rangle\, \mathrm{e}^{-\mathrm{i}\omega t - \kappa t/2} \tag{7.11}$$

(2) From the solution obtained in item (1), compute the equal-time commutator

$$\Xi(t) = \left[a(t), a^\dagger(t)\right] \tag{7.12}$$

and show that $\Xi(t) = 1$ at all times provided that

$$\langle \left[\xi(t), a^\dagger(0)\right] \rangle = 0 \quad \text{for } t > 0\,, \qquad \int\limits_0^t \mathrm{d}t'\, \delta(t - t') = \frac{1}{2} \tag{7.13}$$

This relation is often interpreted in terms of causality: "The past of the system is not influenced by future influences." The main message to keep in mind is: "Quantum noise is fed into an open system to maintain its commutators at the level required by basic quantum mechanics."

(3) Repeat the exercise for the correlation function

$$C_a(\tau) = \lim_{t\to\infty} \langle a^\dagger(t + \tau)a(t) \rangle \tag{7.14}$$

and show that one gets the same Lorentzian spectrum as in the lecture where we used the regression formula. You may have to generalize the 'causality argument' mentioned before.