Makroskopische Quantenzustände und kalte Gase

Markus Gühr / Carsten Henkel (Sommer-Semester 2018)

Problem Set No 5 Date: 25 June 2018

Discussion: 10 July 2018

Problem 5.1 – Gross–Pitaevskii equation (8 points)

In the lecture, we have seen the nonlinear Schrödinger equation for the 'condensate wave function' ϕ

$$-\frac{\hbar^2}{2m}\nabla^2\phi + V(\mathbf{r})\phi + g|\phi|^2\phi = \mu\phi \tag{5.1}$$

where the interaction parameter $g = 4\pi\hbar^2 a/m$ is proportional to the scattering length a. In this problem, we assume that ϕ is a real function and that its integral is equal to the number N of condensed atoms.

(1) Show that ϕ minimises the following 'energy functional'

$$E[\phi] = \int \mathrm{d}^3r \left[\frac{\hbar^2}{4m}|\nabla\phi|^2 + \frac{1}{2}\left(V(\mathbf{r}) - \mu\right)\phi^2 + \frac{g}{4}\phi^4\right] \tag{5.2}$$

(Use the Euler-Lagrange equations of variational calculus.)

(2) Consider a harmonic potential $V(\mathbf{r}) = \frac{k}{2}\mathbf{r}^2$, make a Gaussian *Ansatz* for ϕ, compute the energy functional $E[\phi]$ by solving the Gaussian integrals and minimise the energy by choosing suitable parameters for the Gaussian. This energy minimum is an estimate for the chemical potential (an upper bound? a lower bound?): plot its value as a function of the interaction parameter g and the atom number N.

(3) Make a sketch of $E[\phi]$ as a function of the width of the Gaussian considered in question (2). Now, consider the case $g < 0$ where the atom-atom interactions are attractive. What scenarios would you expect? (keyword: 'Bosenova')

Problem 5.2 – Feeling time of flight (6 points)

A typical procedure to measure the temperature of ultracold atoms is the 'time of flight' technique. It is very simply: any fields that trap the atoms are switched off, the atoms fall in Earth's gravitational field and expand.

(1) Take a typical above-BEC temperature of $0.5\,\mu\mathrm{K}$, a spatial size of $100\,\mu\mathrm{m}$, and your favorite alkali atom. After how much time of flight is the broadening of the cloud due to the initial velocity width larger than the initial spatial width? (You may consider for simplicity the motion only in the horizontal plane.) How far did the atoms

fall during that time? (If this number is too big, you have to invent an anti-gravity field—any suggestions?)

(2) Traps for ultracold atoms are often anisotropic. A typical ratio between the trap frequencies in an elongated trap is $\sqrt{8}$. Consider the ground state of the trap and give the ratios between (a) its spatial widths and (b) its velocity widths. Describe what happens to the size and shape of the cloud during time of flight, provided all atoms are in the trap ground state. A typical trap frequency is in the $100\,\mathrm{Hz}$ range. What would the expansion look like in the extreme case of a quasi-1D trap: 'slow' frequency $10\,\mathrm{Hz}$ and 'strong confinement' by $10\,\mathrm{kHz}$? Your apparatus gives you a height of free fall of approx. $5\,\mathrm{cm}$.

Problem 5.3 – Superconductors: Ginzburg–Landau model (6 points)

A simple model for superconductors is the Ginzburg–Landau equation

$$-\frac{\hbar^2}{2m}\left(\nabla + \frac{2\mathrm{i}e}{\hbar}\mathbf{A}\right)^2 \psi + 2eU(\mathbf{r})\psi + g|\psi|^2\psi = \mu\psi \tag{5.3}$$

where U and $\mathbf{A}$ are the electromagnetic potentials and $2e$ is the charge of the 'macroscopic Cooper pair field' ψ. The density of superconducting electrons is $n = |\psi|^2$.

(1) In a homogeneous system and in zero field, show that the superconducting density is μ/g. (In Ginzburg–Landau theory, g is actually temperature-dependent.)

(2) By replacing μ with the time derivative $\mathrm{i}\hbar\partial_t$, one gets the time-dependent Ginzburg–Landau equation. Show that the following continuity equation holds (please double check prefactors and signs):

$$\partial_t n + \nabla\cdot\mathbf{j} = 0\,, \qquad \mathbf{j} = \frac{\hbar}{m}\,\mathrm{Im}(\psi^*\nabla\psi) - \frac{2e}{m}\mathbf{A}|\psi|^2 \tag{5.4}$$

The second part of the current is called 'diamagnetic'.

(3) Superconductors show the Meißner effect: magnetic fields are expelled from their interior. As a simple model, consider a solution ψ to the stationary Ginzburg-Landau equation whose phase is spatially constant ('the order parameter is rigid'). Argue that in a stationary field, the Maxwell-Ampère equation reduces to

$$\nabla\times\mathbf{B} = -\frac{4e^2 n\mu_0}{m}\mathbf{A} \tag{5.5}$$

Show that this equation allows for exponentially decaying solutions with the penetration depth

$$\xi = \sqrt{\frac{m}{4e^2 n\mu_0}} \tag{5.6}$$

Typical numbers: $n \sim 10^{22}\,\mathrm{cm}^{-3}$, $m = m_e$.