Mathematical Bits
- Group Theory for Physicist -

Problem Set No 1 ($20 + \pi$ Scores)[1]

Emission 15.04.19 – Absorption 03.05.19 – Digestion 06.05.19

▷ **Aufgabe 1 (Yes or No)** (1 Punkt)

Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ be the set of the natural numbers (incl 0). Somebody claims $(\mathbb{N}_0, +, 0)$ constitutes a group. You

- ☐ agree
- ☐ don't agree

▷ **Aufgabe 2 (Dimension of Matrix Groups (I))** (1 Punkt)

The dimension of the General Linear Group $GL(n, \mathbb{C})$ is given by

- ☐ n, since these are n-dimensional matrices
- ☐ n^2, since these have n^2 elements
- ☐ $2n^2$, since there is real and imaginary part

▷ **Aufgabe 3 (Dimension of Matrix Groups (II))** (2 Punkte)

The $SO(n)$ is the set of all real $n \times n$ matrices with unit determinant whose transpose equals the inverse.

Show that $SO(n)$ with matrix multiplication and neutral element the unit matrix constitutes a group of dimension $n(n-1)/2$.

▷ **Aufgabe 4 (Dimension von Matrixgruppen (III))** (2 Punkte)

The $SU(n)$ is the set of all complex $n \times n$ matrices with unit determinant whose adjoint equals the inverse.

Show that $SU(n)$ with matrix multiplication and neutral element the unit matrix constitutes a group of dimension $n^2 - 1$. For which pairs m, n the dimension of $SU(m)$ equals that of $SO(n)$?

▷ **Aufgabe 5 (Groups of order 3)** (2 Punkte)

Show that there is exactly one abstract group of order 3, isomorphic the cyclic group C_3. Provide some realisations.

▷ **Aufgabe 6 (Groups of order 4)** (3 Punkte)

Show that there are exactly two abstract groups of order 4, one isomorphic C_4, the other isomorphic $V_4 \cong C_2 \otimes C_2$, called *Klein'sche Vierergruppe*.

▷ **Aufgabe 7 (Gruppen der Ordnung n)** (3 Punkte)

Show that each group G with prime order is isomorph the cyclic group C_n, i.e. $G \cong C_n$.

[1] Aufgaben mit transzendenter Punktezahl sind fakultative Nüsse. Nüsse sind bekanntlich nahrhaft ...

▷ Aufgabe 8 (Euler Theorem) (6 Punkte)

Euler's Theorem states that each rotation matrix $\underline{R} \in SO(3)$ has an eigenvector to eigenvalue 1. Accordingly, all vectors $\underline{v}$, which are multiples of this eigenvector, are invariant under $\underline{R}$, i.e. $\underline{R}\,\underline{v} = \underline{v}$, which – in geometrical terms – implies , that $\underline{R}$ specifies a straight line – the axis of rotation.

(a) Recall that λ eigenvalue of a matrix $\underline{A}$ equivalent $\det(\underline{R} - \lambda\underline{1}) = 0$. Prove Euler's Theorem by showing $\det(\underline{R} - \underline{1}) = 0$.
Hint: Exploit orthogonality of $\underline{R}$ and recall the properties of the determinant.

(b) Does Euler's Theorem hold for $SO(2)$, or even more general for $SO(2n)$?

▷ Aufgabe 9 (Rubik's Cube) (π Punkte)

Rubik's Magic Cube was quite popular a puzzle in the 1980's. Consult the Wikipedia if you are not familiar with it. Here a short quote,[2]

> Ideal Toy Company stated on the package of the original Rubik cube that there were more than three billion possible states the cube could attain. It's analogous to Mac Donald's proudly announcing that they've sold more than 120 hamburgers. (J. A. Paulos, Innumeracy)

and our kind request to estimate the number of possible configurations of the cube.

[2] gefunden auf `http://www.gap-system.org/Gap3/Doc3/Examples3/rubik.html`