Theoretische Physik III
- Quantenmechanik (SoSe 2019) -

Problemset 02 ($20 + \pi$ Punkte)[1]

Emission 15.04.18 – Absorption 23.04.18 – Digestion tba

Problems with asterix are isomorphic to problems of the exam

▷ **Aufgabe 1 (Particle in a box)*** (8 Scores)

A particle moves freely within the limits set by the walls of a cubic box of edge length L.

(a) Determine the energy levels and eigenfunctions. Show that the energy levels are given by

$$E_{klm} = \epsilon \left((l+1)^2 + (m+1)^2 + (n+1)^2\right) , \qquad l, m, n = 0, 1, 2, \ldots , \tag{1}$$

with $\epsilon = \hbar^2\pi^2/(2mL^2)$, and corresponding eigenfunctions

$$\varphi_{klm}(x, y, z) = \left[\frac{2}{L}\right]^{\frac{3}{2}} \sin(k_l x)\sin(k_m y)\sin(k_n z) , \qquad k_l = \frac{(l+1)\pi}{L} \text{ etc} , \tag{2}$$

with the box placed with the front left corner in the origin of our cartesian coordinate system.

(b) With the particle in the ground sate – what pressure does it exert onto the walls of the box?
Recall: "Pressure" equals "Force-per-Area", where "Force" equals "Work-per-Displacement", and "Work" is a form of transfer of "Energy". So – just determine the change of the ground state energy which comes with a small displacement of one of the walls.

(c) How large a value could $\hbar$ possibly assume in order not to be killed by the seeds in opening an off-the-mill melon? As a theoretical physicist you may assume that melons are cubicle shaped – what they are indeed, as the figure below indicates.

(d) convince yourself that (1) the spacing of the energy levels is the more densly the larger the box, and (2) the larger the energy the more levels in its neighborhood. In the limit $L \to \infty$ the energy spectrum is "quasi-continuous". Derive the density of states – that is the number of levels whose energies fall into an interval dE at E.

[1] Problems with transcendental scores are facultative nuts. Nuts come high nutrition value . . .

▷ **Aufgabe 2 (Some Theorem)** (3 Scores)

Let $\hat{T}$ be a linear operator on some Hilbert space $\mathcal{H}$, and $\hat{T}^\dagger$ its adjoint. Prove the following inequality

$$\langle \hat{T}^\dagger \hat{T} \rangle_\psi \geq 0\,, \qquad \forall \psi \in \mathcal{H}\,. \tag{3}$$

▷ **Aufgabe 3 (Uncertainty relations)** (4 Scores)

Recall the variance (uncertainty) of an observable in state ψ, $\delta_\psi A := [\langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle_\psi]^{1/2}$.

For $\hat{A}$ and $\hat{B}$ selfadjoint with commutator

$$[\hat{A}, \hat{B}] = i\hat{C}\,. \tag{4}$$

prove the following inequality

$$\delta A_\psi \delta_\psi B \geq \frac{1}{2} |\langle \hat{C} \rangle_\psi|\,. \tag{5}$$

<u>Hint:</u> Exploit Problem No 2 with the replacement $\hat{T} = \hat{A} - \langle \hat{A} \rangle + i\lambda(\hat{B} - \langle \hat{B} \rangle)$ and minimize with respect to λ.

▷ **Aufgabe 4 (Minimal Uncertainty State)** (5 Scores)

For a point particle in $\mathbb{R}$ with canonical commutator $[\hat{q}, \hat{p}] = i\hbar$ problem No 3 turn into *Heisenberg Uncertainty Relation*,

$$\delta q \delta p \geq \frac{\hbar}{2}\,. \tag{6}$$

A state for which equality holds is dubbed "minimum uncertainty state". Show that the most general state with minimum uncertainty is described by a Gaussian in the position representation.

<u>Hinweis:</u> Betrachte Beweis zu Aufgabe 3. Setze o.B.d.A. $\langle \hat{q} \rangle = \langle \hat{p} \rangle = 0$; minimal heißt dann neben $\lambda = \hbar/(2\delta p^2)$ auch $\langle \hat{T}^\dagger \hat{T} \rangle) = 0$, also $\hat{T}\psi_{\text{min}} = 0$. Auswertung dieser Gleichung in Ortsdarstellung (wo $(\hat{q}\psi)(x) = x\psi(x)$ und $(\hat{p}\psi)(x) = \frac{\hbar}{i}\psi'(x)$) liefert den gesuchten Beweis.

▷ **Aufgabe 5 (Quantum diffusion)** (π Scores)

Your friend is in worry. Having a bunk bed he is afraid of quantum diffusion which would make him fall out of bed so that he awakes in morning on the floor, possibly with bruises all over his body.

(a) Try to calm you friend down.
<u>Hint:</u> Model your friend a Gaussian wave packet. Make use of the relation $m\Delta v^2/2 \sim k_\text{B}T$, which you will learn in the course on statistical physics, in order to relate the initial velocity uncertainty δv of your friend, who is of mass m, with his temperature T (k_B is the Boltzmann constant).

(b) Wie lange müsste Ihr Freund gewohnheitsmäßig schlafen, um im Mittel jedes zweite mal neben seinem Bett aufzuwachen?

(c) Contemplate to what extent your answer to (b) appears realistic. If you conclude "unrealistic" – what could be the reason?