Theoretische Physik III
- Quantenmechanik (SoSe 2019) -

Problemset 03 ($20 + \pi$ Punkte)[1]

Emission 23.04.19 – Absorption 30.04.19 – Digestion tba

Problems with asterix are isomorphic to problems of the exam

▷ Aufgabe 1 (Operator Rules) (5 Scores)

Confirm the folloting rules of operator algebra. Questions conerning the domain of definition may be ignored ...

$$(\alpha\hat{A})^\dagger = \alpha^*\hat{A}^\dagger \tag{1}$$

$$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger \tag{2}$$

$$[\hat{A},\hat{B}]^\dagger = [\hat{B}^\dagger,\hat{A}^\dagger] \tag{3}$$

$$(\hat{A}\hat{B})^{-1} = \hat{B}^{-1}\hat{A}^{-1} \tag{4}$$

$$(\hat{A}^\dagger)^{-1} = (\hat{A}^{-1})^\dagger \tag{5}$$

▷ Aufgabe 2 (Initiationsritus Quantenmechanik) (3 Punkte)

Ever since the dawn of mankind, students of quantum mechanics are kindly asked to prove

(a) In unitary space the *Schwarz Inequality* holds[2]

$$|\langle\psi,\chi\rangle| \le \|\psi\|\|\chi\|\,, \tag{6}$$

(b) the *Parallelogram Equality*

$$\|\psi+\chi\|^2 + \|\psi-\chi\|^2 = 2\|\psi\|^2 + 2\|\chi\|^2\,. \tag{7}$$

(c) and the inner product can be expressed

$$4\langle\psi,\chi\rangle = \|\psi+\chi\|^2 - \|\psi-\chi\|^2 + i\|i\psi+\chi\|^2 - i\|i\psi-\chi\|^2\,. \tag{8}$$

▷ Aufgabe 3 (Particle on the circle I)* ($6 + e$ Punkte)

Consider a particle which moves freely on a circle of circumference a. The particle's coordinate q is periodic with period interval $[0, a]$. With m the particle mass, the Hamiltonian reads

$$H = \frac{p^2}{2m} \tag{9}$$

For the following it may be useful to derive the classical equation of motion and solve these for general initialconditions (watch out for the periodicity of the coordinate q).

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[1]Aufgaben mit transzendenter Punktezahl sind fakultative Nüsse. Nüsse sind bekanntlich nahrhaft ...

[2]Für den "technischen Jargon" vgl. die Handreichung zu Hilberträumen, Operatoren etc. auf der Netzseite des Kurses ...

The quantum version of our system follwos the canonical recipe. The position operator $\hat{q}$, defined Die quantisierte Version unseres Systems erhält man in der sog "kanonischen Weise". Der $(\hat{q}\psi)(x) = x\psi(x)$, and canonical momentum operator $\hat{p}$, defined $(\hat{p}\psi)(x) = \frac{\hbar}{i}\psi'(x)$, obey the Heisenberg commutation rule

$$[\hat{q}, \hat{p}] = i\hbar \,. \tag{10}$$

Hilbertspace of our system is the space of square integrable complex funktions on $[0, a]$, i.e. $\mathcal{H} = \{\psi \in \mathcal{L}^2([0, a], dx) | \psi(0) = \psi(a)\}$. The Hamiltonian is given by (9) but with H and p topped with a hat.

(a) Please show: The stationary Schrödingerequation $\hat{H}\psi = E\psi$ on $\mathcal{H}$ is solved

$$E_n = \frac{\hbar^2 k_n^2}{2m} \,, \qquad \varphi_n(x) = \frac{1}{\sqrt{a}} e^{ik_n x} \tag{11}$$

withe wavenumber k_n

$$k_n = \frac{2\pi}{a} n \,, \qquad n = 0, \pm 1, \ldots \,. \tag{12}$$

(b) The φ_n form a complete orthonormal system in $\mathcal{H}$, that is

$$\langle \varphi_m, \varphi_n \rangle = \delta_{mn} \,, \tag{13}$$

$$\forall_{\psi \in \mathcal{H}} \|\psi\|^2 = \sum_n |\langle \varphi_n, \psi \rangle|^2 \,. \tag{14}$$

(c) What are the eigenvalues and eigen functions of the momentum operator?

(d) The momentum operator is not bounded on $\mathcal{H}$, and according to a theorem of functional analysis it is not continuous. Sounds horrible, but is no desaster. Much more important would be whether $\mathcal{H}$ comes with a dense set of functions on which the momentum operator would be self adjoint. Show that this is the case indeed, and construct a corresponding set. (e Scores)

▷ **Aufgabe 4 (Teilchen auf dem Kreis II)*** (6 Punkte)

The particle from the previous problem is prepared, at time $t = 0$, in a state

$$\Psi(x, t = 0) = \alpha\varphi_0(x) + \beta\varphi_1(x) + \gamma\varphi_{-1}(x) \tag{15}$$

with $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. Is Ψ correctly normalized?

(a) What's the meaning of all these coefficients α, β, γ?
<u>Hint:</u> Think in terms of measurements of energy, momentum, or position. Complete the gaps in the following phrases: 1. "With probability [gap] the value [gap] will be read off from an energy measurement device"; 2. "With probability [gap] the value [gap] will be read off from a momentum measurement device"; 3. "With probability [gap] the value [gap] will be read off from a position measurement device, that queries the presence in dx at x_0"?

The state Eq. (15) evovles according to the Schrödinger equation $i\hbar\dot{\Psi} = \hat{H}\Psi$. Subsequently, at time $t = T$, a measurement is executed

(b) Complete the gaps in the phrases of (a). For which measurements the results do no depend on T? Why don't they?