Theoretische Physik III
- Quantenmechanik (SoSe 2019) -

Übungsblatt 05 (20 Punkte)

Ausgabe 06.05.19 – Abgabe 14.05.19 – Besprechung n.V.

Aufgaben mit Sternchen sind Klausurisomorph

▷ Aufgabe 1 (Connection formulae for δ-potential)* (2 Scores)

For the stationary Schrödinger equation $E\psi = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + b\delta(x-a)\right]\psi(x)$, with δ the delta function, please derive the connection formula at $x = a$,

$$\psi'(a_+) - \psi'(a_-) = \frac{2mb}{\hbar^2}\psi(a)\,, \tag{1}$$

where $\psi'(a_\pm) = \lim_{\varepsilon\to 0_+} \frac{d\psi}{dx}|_{x=a\pm\varepsilon}$.

<u>Hint:</u> You may want to integrate the stationary Schrödinger equation, $\int_{a-\varepsilon}^{a+\varepsilon}$. Remember that ψ is finite and continuous, even at $x = a$.

▷ Aufgabe 2 (Kronig-Penney Modell) (10 Punkte)

An electron in a metal experiences a periodic potential. Despite its motion being unlimited, because of the potential's periodicity there are only certain energy bands accessible to the electron.

(a) Derive the electronic band structure for the Kronig-Penney model

$$V(x) = \alpha \sum_{j=-\infty}^{+\infty} \delta(x - ja) \tag{2}$$

<u>Hinweis:</u> You may want to recall the Bloch Theorem. The Bloch function is to be derived on the interval $(0, a]$. You are then faced by only one delta function whcih sits at the right end of the interval. You deal with this one using the connection formula from above.

(b) Picture the position probability of the electron for stationary states at the upper and lower band edge respectively for the case of an 'ttractive" potential, $\alpha < 0$.

▷ Aufgabe 3 (Coherent states) (8 Punkte)

In the lecture on the harmonic oscillator you were familiarized with the eigenvectors of $\hat{a}^\dagger\hat{a}$, the so called *Fock states* $|n\rangle$, defined $\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$ (in position representation $\varphi_n(x) = \langle x|n\rangle$). Fock states, you recall, are the stationary states of the harmonic oscillator.

With stationary states nothing is in motion. But for the harmonic oscillator you want to see some swinging. In order to see "swinging" in the quantum mechanical oscillator one has to study the time evolution of linear superpositions of Fock states. And a particulary important class of such linear superpositions are the so called *coherent states*,

$$|\alpha\rangle := e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,, \tag{3}$$

with $\alpha \in \mathbb{C}$ a complex number which lables the state. Please show

(a) The coherent state $|\alpha\rangle$ is an eigenvector of the annilhilation operator to eigenvalue α,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{4}$$

In what follows we shall use suitable units for position $\hat{q}$ and momentum $\hat{p}$, such that $\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p})$ with $[\hat{q}, \hat{p}] = i$. Please show

(b) expectation values of position and momentum for coherent states $|\alpha\rangle$ read

$$\langle \hat{q} \rangle = \frac{1}{\sqrt{2}}(\alpha + \alpha^*) \,, \tag{5}$$

$$\langle \hat{p} \rangle = \frac{i}{\sqrt{2}}(\alpha^* - \alpha) \,, \tag{6}$$

(c) $|\alpha\rangle$ is a state of minimal uncertainty, $\delta_\alpha q \delta_\alpha p = 1/2$.

(d) the position representation of $|\alpha\rangle$, namely $\psi_\alpha(x) := \langle x|\alpha\rangle$, is nothing but a Gaussian, centered at $\langle q\rangle$, of width $1/\sqrt{2}$, and phase factor $e^{i\langle\hat{p}\rangle x}$.

<u>Hinweis:</u> Recall how the ground state position representation was derived .

(e) We now study the dynamics of the harmonic oscillator coherent states. At time $t = 0$ the ocillator is prepared in a coherent state $|\alpha\rangle$. Please show that at any later instance, the oscillator is still in a coherent state. Derive the amplitude $\alpha(t)$. Picture $\alpha(t)$ on the complex plane. Also picture $|\langle x|\alpha(t)\rangle|$. Appreciate the emergent picture of the swinging particle. Realize, that the complex α-plane is intimately related to the classical phase space.

In electrodynamics / quantum optics “position” and “momentum” are called quadrature amplitudes, with “position” being substituted by electric field strength, and “momentum” its time derivative. The operator $\hat{n} := \hat{a}^\dagger \hat{a}$ is associated the photon number operator. Please show:

(f) in a coherent state the statistics of the photon number is a Poissonian,

$$P(n) \equiv |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} |\alpha|^{2n}/n! \,; \tag{7}$$

(g) Expectation value and square variance of the photon number in a coherent state are given by

$$\langle \hat{n} \rangle = |\alpha|^2 \tag{8}$$

$$\delta_\alpha^2 n = |\alpha|^2 \tag{9}$$