Theoretische Physik III
- Quantenmechanik (SoSe 2019) -

Übungsblatt 08 ($20 + \pi$ Punkte)[1]

Ausgabe 27.05.19 – Abgabe 04.06.19 – Besprechung n.V.

Aufgaben mit Sternchen sind Klausurisomorph

▷ Aufgabe 1 (Spin Matrices) (4 Punkte)

Prove that the so-called spin matrices

$$\hat{s}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \quad \hat{s}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \quad \hat{s}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \tag{1}$$

obey the angular momentum algebra. What would be their relation to the representation of the anbular momentum operators on the vector space (!) of spherical harmonics $Y_{\ell m}$ for quantum number $\ell = 1$?

▷ Aufgabe 2 (More Spinology . . .)* (6 Punkte)

The rotation of the preparation device for the spin polarization of spin-$\frac{1}{2}$ particles are described by a unitary operator,

$$\hat{U}_{\phi\vec{n}} := \exp\left\{ -\frac{i}{\hbar}\phi\vec{n}\cdot\hat{\vec{s}} \right\} \tag{2}$$

with $\vec{n}$ the Euclidean unit vector in direction of the axis of rotation, ϕ the angle of rotation, and $\hat{\vec{s}}$ spin vetor operator for spin-$\frac{1}{2}$ particles.

(a) What is the expression of $\hat{U}$ in the standard matrix representation?

<u>Hinweis:</u> Recall the series respresenation of the exponential function, and $e^{ix} = \cos(x) + i\sin(x)$ . . .

(b) Compute $\hat{U}^\dagger_{\phi\vec{n}}\hat{\vec{s}}\hat{U}_{\phi\vec{n}}$. Convince yourself that $\hat{\vec{s}}$ is quite rightfully called *vector*-operator.

▷ Aufgabe 3 (Addition von Bahndrehimpuls und Spin-$\frac{1}{2}$) (12 Punkte)

For the electron of atomic hydrogen, spin included, the total angular momentum ist just the sum of orbital angular momentum and spin,

$$\hat{\vec{j}} := \hat{\vec{l}} + \hat{\vec{s}}. \tag{3}$$

The common eigenstates to $\hat{\vec{j}}^2$, $\hat{j}_z$, $\hat{\vec{\ell}}^2$ and $\hat{\vec{s}}^2$ are written $|jm_j\ell s\rangle$, the entries separated by a comma, if necessary, with quantum numbers j, m_j, ℓ and s, by definition

$$\begin{aligned} \hat{\vec{j}}^2|jm_j\ell s\rangle &= \hbar^2 j(j+1)|jm_j\ell s\rangle , & \hat{j}_z|jm_j\ell s\rangle &= \hbar m_j|jm_j\ell s\rangle , \\ \hat{\vec{\ell}}^2|jm_j\ell s\rangle &= \hbar^2\ell(\ell+1)|jm_j\ell s\rangle , & \hat{\vec{s}}^2|jm_jls\rangle &= \hbar^2 s(s+1)|jm_jls\rangle , \end{aligned} \tag{4}$$

[1] Aufgaben mit transzendenter Punktezahl sind fakultative Nüsse. Nüsse sind bekanntlich nahrhaft . . .

In the following $s = \frac{1}{2}$, and ℓ ist variabel $\ell = 0, 1, 2, \ldots$. To each ℓ (with exception $\ell = 0$) we have two possible values $j = \ell \pm \frac{1}{2}$. For $\ell = 0$ we simply have $j = \frac{1}{2}$.

The goa is to express the $|jm_j ls\rangle$ as a linear combination of the $|\ell m_\ell; s\mu\rangle := |\ell m_\ell\rangle \otimes |s\mu\rangle$, with quantum numbers m_ℓ and μ by definition $\hat{\ell}_z|\ell m_\ell s\mu\rangle = \hbar m_\ell |\ell m_\ell s\mu\rangle$, $m_\ell = -\ell, -\ell + 1, \ldots, \ell$, und $\hat{s}_z|\ell m_\ell s\mu\rangle = \hbar\frac{\mu}{2}|\ell m_\ell s\mu\rangle$ mit $\mu = \pm 1$. In each case $m_j = -j, -j+1, \ldots, j$.

Show that for $\ell = 1, 2, \ldots$

$$|\ell \pm \tfrac{1}{2}, m_j; \ell, \tfrac{1}{2}\rangle = \sqrt{\frac{\ell + \frac{1}{2} + m_j}{2\ell + 1}}|\ell, m_j \mp \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\pm\rangle \pm \sqrt{\frac{\ell + \frac{1}{2} - m_j}{2\ell + 1}}|\ell, m_j \pm \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\mp\rangle \quad (5)$$

and for $\ell = 0$

$$|\tfrac{1}{2}, \pm\tfrac{1}{2}; 0\tfrac{1}{2}\rangle = |0, 0\rangle \otimes |\tfrac{1}{2}\pm\rangle\,. \quad (6)$$

In spectroscopic notation the m_j-multiplets are coded $n\ell_j$, for example $2p_{\frac{1}{2}}$ or $2p_{\frac{3}{2}}$. In the gross structure (“Kepler-Atom”) these two levels are energetically degenerate. Upon inclusion of the spin-orbit interaction (fine structure), the degeneracy is lifted …

▷ **Aufgabe 4 (Wohnst-Du-noch)** (π Punkte)

You ordered some spin-1/2 particles in the internet. Unpacking, you realize that the the manual is misson which would specify the spin state of the particles. So, your job is to design a procedure which would reveal the spin state. At your disposal a Stern-Gerlach magnet with variable orientation.