Theoretische Physik III
- Quantenmechanik (SoSe 2019) -
Übungsblatt 09 (20 Punkte)
Ausgabe 03.06.19 – Abgabe 11.06.19 – Besprechung n.V.
Aufgaben mit Sternchen sind Klausurisomorph

▷ **Aufgabe 1** Ritz Theorem] (6 Punkte)

Prove the Ritz Theorem, which states that the functional is stationary $E[\psi] = \langle\psi|\hat{H}|\psi\rangle/\langle\psi|\psi\rangle$, that is $\delta E[\psi] = 0$, if and only if $\psi = \psi_0$ eigen vector of $\hat{H}$, i.e $\hat{H}\psi_0 = E_0\psi_0$. Conclude from the Ritz theorem $E[\psi] \geq E_0$, with E_0 the ground state energy. Browse some textbook for applications ...

▷ **Aufgabe 2 (Ground state energy via Ritz)*** (6 Punkte)

Using the Ritz theorem, pleas estimate the ground state energy of the electron in the Coulomb field of a Z-fold charged nucleus. Use $\propto e^{-\kappa r}$ as a variational ansatz, with κ variation parameter. How does your result compare with the exact result?

▷ **Aufgabe 3 (Hyperfine structure)** (8 Punkte)

[Was die HFS ist. und wo sie herkommt, sollte man wissen ...]

In the hyper fine structure results from the interaction of the electron spin with the nuclear spin. In the case of atomic hydrogen the proton magnetic moment, $\vec{\mu}_\mathrm{p} = \gamma_\mathrm{p}\vec{s}_\mathrm{p}$, $\gamma_p \approx 2,79e_0/m_\mathrm{p}$, gives rise to a magentic field

$$\vec{B}_\mathrm{p}(\vec{x}) = -\frac{\mu_0}{4\pi r^3}\left(\vec{\mu}_\mathrm{p} - 3\frac{(\vec{\mu}_\mathrm{p}\cdot\vec{x})\vec{x}}{r^2}\right) + \frac{2\mu_0}{3}\vec{\mu}_\mathrm{p}\delta(\vec{x})\,. \tag{1}$$

under the assumption, that the proton is placed in the origin, and $r = |\vec{x}|$.

Die Einstellenergie des magnetischen Moments des Elektrons, $\mu_\mathrm{e} = -\gamma_\mathrm{e}\vec{s}_\mathrm{e}$, $\gamma_\mathrm{e} = e_0/m_\mathrm{e}$ (Annahmen: $g = 2$), lautet

$$\hat{H}_\mathrm{HFS} = -\hat{\vec{\mu}}_\mathrm{e}\cdot\vec{B}_\mathrm{p}(\hat{\vec{q}}) \tag{2}$$

In order to estimate the impact of the hyper-fine structure interaction on the hydrogen groundstate, we treat $\hat{H}_\mathrm{HFS}$ for the translational degrees of freedom in forst order perturbation theory, but keep the full expression for the spin degrees of freedom. Averaging the proton-spin induced magentic field with weight function $|\psi_{100}(\vec{x})|^2$ only the contact-term contributes (why?), thus

$$\hat{H}_\mathrm{HFS} = -\frac{2\mu_0}{3}\hat{\vec{\mu}}_\mathrm{e}\hat{\vec{\mu}}_\mathrm{p}|\psi_{100}(0)|^2 = \frac{A}{\hbar^2}\hat{\vec{s}}_\mathrm{e}\cdot\hat{\vec{s}}_\mathrm{p} \tag{3}$$

with

$$A = \frac{16}{3}\times 2,79\frac{m_\mathrm{e}}{m_\mathrm{p}}\alpha^2 E_\mathrm{Ry} \approx 5,87\times 10^{-6}\mathrm{eV}\,. \tag{4}$$

or $\nu = A/h \approx 1417\mathrm{MHz}$ and $\lambda = c/\nu \approx 21\mathrm{cm}$, respectively.

(a) Show that the eigenproblem of the interaction Hamiltonian (3) is solved by

$$\begin{aligned} E_+ &= E_0 + A/4 \qquad && \text{im Triplet } |S=1,M\rangle,\ M=-1,0,1\,, \\ E_- &= E_0 - 3A/4 \qquad && \text{im Singlet } |S=0,M=0\rangle\,. \end{aligned} \tag{5}$$

(b) In an external magnetic field $\vec{B} = B\vec{e}_z$ the triplet splits, and the singlet is shifted. Compute the split and shift as function of the magnetic field strength (you may neglect the coupling of the proton spin to the external field - why?). Plot the energy values as function of the magentic field strength, and identify the weak field Zeeman-regime and the strong field Paschen-Back regime.

Remark: The ground state haperfine structure plays an important role in astrophysics (21cm-Linie), and is frequently used in test of the general theory of relativity (gravitative redshift). Besides, the hyperfine transition of the Cs-133 isotope serves for the definition of a second (1 second equals 9 192 631 770 oscillations between hyperfine levels). Because the transition is dipole forbidden, it is extremely weak with an average lifetime of $\sim 3,5 \times 10^{14}$sec $\sim 10^7$Jahre, with the transition being due to magentic dipole and electric quadrupole transitions.

An finally - the plate carried by the Pioneer 10 Mission depicts the hyperfine structure of atomic hydrogen in order communicate a time and length standard to other civilisations out there ...